Real Arthur packets from a sheaf theoretic perspective

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Classical groups - L-groups - L-parameters

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- Let $W_\mathbb{R}$ be the **Weil group** of $\mathbb{R}$:

$$W_\mathbb{R} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$$

in which

$$j^2 = -1 \quad \text{and} \quad jzj^{-1} = \bar{z},\ z \in \mathbb{C}^\times.$$

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is commutative.

ii. for every $w \in W_{\mathbb{R}}$ the projection of $\varphi(w)$ onto ${}^\vee G$ is semisimple.

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$$\Pi(G(\mathbb{R}, \sigma)) \quad = \quad \bigsqcup_{\varphi \in \Phi(G)} \Pi_\varphi(\sigma).$$

The sets $\Pi_\varphi(\sigma)$ *are called* ***L*-packets**.

Local Arthur Conjectures : the real case

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$$\Psi(G) \ = \ \underline{A \text{ - } \textbf{parameters}} \text{ of } G,$$

consisting of ${}^\vee G$-equivalence classes of group homomorphisms

$$\psi : W_{\mathbb{R}} \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow {}^L G,$$

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$$\psi : W_{\mathbb{R}} \times \mathrm{SL}(2,\mathbb{C}) \longrightarrow {}^L G,$$

satisfying :

i. The restriction of ψ to $W_{\mathbb{R}}$ is a **bounded** Langlands parameter.

ii. The restriction of ψ to $\mathrm{SL}(2,\mathbb{C})$ is **holomorphic**.

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Local components of automorphic representations.

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by defining for every $w \in W_\mathbb{R}$,

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Then the **A-packet** of ψ should contain the L-packet of φ_ψ :

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Furthermore, $\Pi_\psi(\sigma)$ should be the support of a **stable virtual character** $\eta_\psi(\sigma)$:

$$\mathrm{Supp}\left(\eta_\psi(\sigma)\right) \ = \ \Pi_\psi(\sigma).$$

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Property B :

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Property B : Define

$$^\vee G_\psi \;=\; \text{the centralizer in } {}^\vee G \text{ of the image of } \psi.$$

and consider its component group :

$$A_\psi \;=\; {}^\vee G_\psi / \left({}^\vee G_\psi\right)_0 .$$

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Then associated to each $\pi \in \Pi_\psi(\sigma)$ there is a non-zero finite-dimensional representation of A_ψ :

$$\begin{aligned} \tau_\psi : \Pi_\psi(\sigma) &\longrightarrow \hat{A}_\psi \\ \pi &\longmapsto \tau_{\psi_G}(\pi), \end{aligned}$$

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such that the stable virtual character of **Property A** satisfies :

$$ \eta_\psi(\sigma) = \sum_{\pi \in \Pi_\psi(\sigma)} \varepsilon_\pi \, \dim(\tau_\psi(\pi)) \, \pi. $$

Here $\varepsilon_\pi = \pm 1$ is to be defined.

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Property C :

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$$\boxed{\textsc{Spectral Identities}}$$

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Property D : The irreducible representations of $\Pi_{\psi}(\sigma)$ are all **unitary**.

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$$\left\{ \begin{array}{c} \text{Harmonic} \\ \text{Analysis} \end{array} \right\}$$

$$\searrow$$

$$\left\{ \begin{array}{c} \text{Local Arthur} \\ \text{Conjectures : real} \\ \text{case} \end{array} \right\}$$

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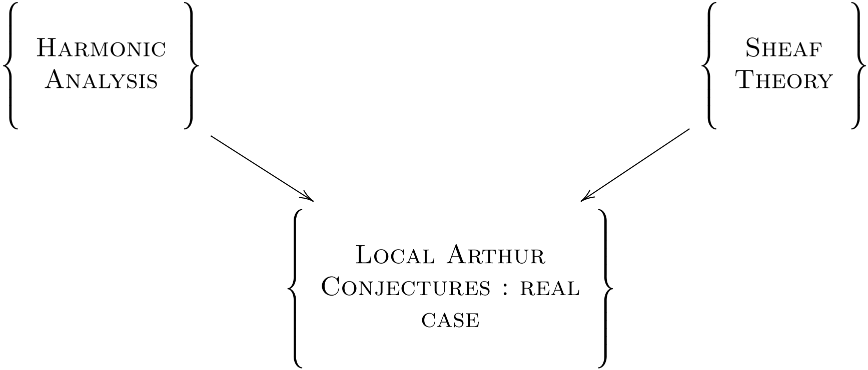

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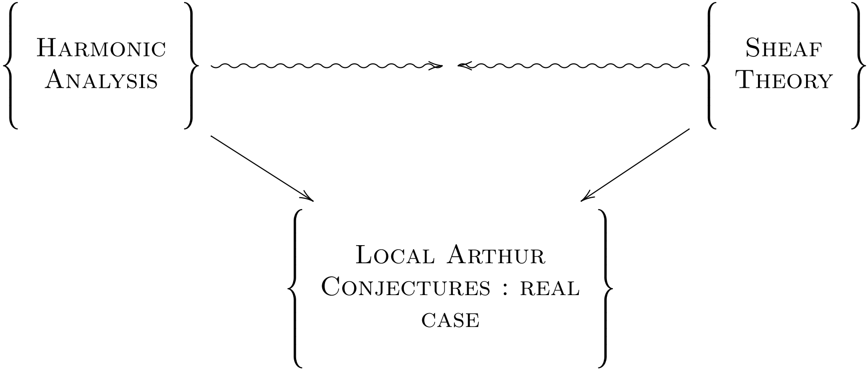

Local Arthur Conjectures : the real case

First Approach : The harmonic analysis approach has been used in :

1. **J. Arthur** (2014)

   "The Endoscopic Classification of Representations",

   to prove the Local Arthur Conjectures in the case of **quasi-split classical groups**.

2. **C.P MOK** (2015)

   "Endoscopic classification of representations of quasi-split unitary groups",

   to prove the Local Arthur Conjectures in the case of **quasi-split Unitary groups**.

3. **C. Moeglin and D. Renard** (2020)

   "Sur les paquets d'Arthur des groupes classiques réels",

   to extend the proof of the Local Arthur Conjectures to all **pure real forms** of **classical and unitary groups**.

Local Arthur Conjectures : the real case

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Their solution to the conjectures is valid in the case of

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and it is based on sophisticated geometric tools :

Microlocal Geometry : $\mathcal{D}$-modules and perverse sheaves, **characteristic cycles**.

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- First, we give a short description of the two different approaches. We will concentrate our exposition on **ABV**'s sheaf theoretic approach.
- Finally, we give an overview of the proof in

"Equivalent definitions of Arthur packets for real classical groups"
J. Adams - A - P. Mezo.

and

"Equivalent definitions of Arthur packets for real unitary groups"
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of the equivalence of the two approaches in the case of pure real forms of classical and unitary groups.

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$$\psi_G \longrightarrow \eta^{\mathrm{Ar}}_{\psi_G}(\sigma_q) \xrightarrow{\mathrm{Supp}} \Pi^{\mathrm{Ar}}_{\psi_G}(\sigma_q)$$

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Once the equality between packets is obtained, it is not difficult to prove

$$\tau^{\mathrm{ABV}}_{\psi} \;=\; \tau^{\mathrm{Ar}}_{\psi}.$$

Arthur's approach

$$\psi_G \longrightarrow \eta^{\mathrm{Ar}}_{\psi_G}(\sigma_q) \longrightarrow \Pi^{\mathrm{Ar}}_{\psi_G}(\sigma_q).$$

Let us begin by discussing the strategy followed by Arthur and Mok to define $\Pi^{\mathrm{Ar}}_{\psi_G}(\sigma_q)$.

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$$H := \mathrm{GL}_N \quad \text{if} \quad G \text{ is a } \textbf{classical group}$$
$$H := \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N = \mathrm{GL}_N \times \mathrm{GL}_N \quad \text{if} \quad G \text{ is a } \textbf{unitary group}.$$

with

$$H(\mathbb{R},\sigma_q) = \begin{cases} \mathrm{GL}_N(\mathbb{R}) & \text{if} \quad G \text{ is a } \textbf{classical group} \\ \mathrm{GL}_N(\mathbb{C}) & \text{if} \quad G \text{ is a } \textbf{unitary group}, \end{cases}$$

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$$\vartheta(g) \ = \ \tilde{J}\,(g^{-1})^{\intercal}\,\tilde{J}^{-1}, \ g \in \mathrm{GL}_N,$$

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The work of **Mezo** and **Shelstad**, proves that **twisted characters** of $H(\mathbb{R}, \sigma_q) \rtimes \vartheta$ are related to stable distributions on $G(\mathbb{R}, \sigma_q)$, through the **(twisted) Transfer map** :

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The underlying novelty in **ABV**'s work is the introduction of a space :

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$$\begin{aligned} \Xi(G) \quad &\longleftrightarrow \quad \Pi(G/\mathbb{R}) = \coprod_{\sigma \text{ in the inner class of } \sigma_q} \Pi(G(\mathbb{R}, \sigma)) \\ \xi \quad &\longleftrightarrow \quad \pi(\xi). \end{aligned}$$

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Since by the **Langlands classification** each $\pi(\xi) \in \Pi(G/\mathbb{R})$ is the Langlands quotient of a unique **standard representation** :

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Since by the **Langlands classification** each $\pi(\xi) \in \Pi(G/\mathbb{R})$ is the Langlands quotient of a unique **standard representation** :

$$\pi(\xi) \;=\; \overline{M(\xi)},$$

the refined correspondence yields also a bijection

$$\xi \quad \longleftrightarrow \quad M(\xi).$$

Adams-Barbasch-Vogan's approach

To each $\xi \in \Xi(G)$ we have attached two objects :

$$\pi(\xi) \longleftarrow \xi \longrightarrow M(\xi).$$

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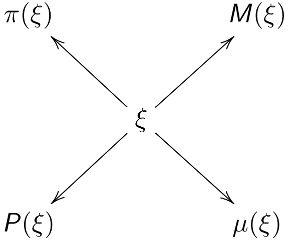
Adams-Barbasch-Vogan's approach

To each complete geometric parameter $\xi \in \Xi(G)$ we have attached four objects :



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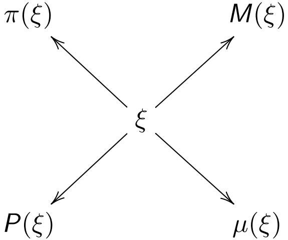

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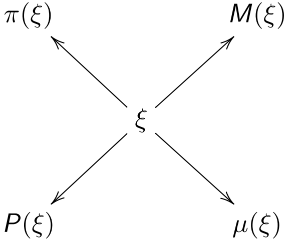

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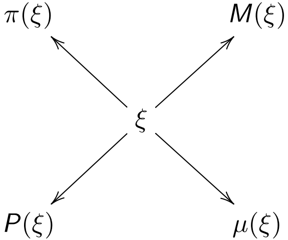

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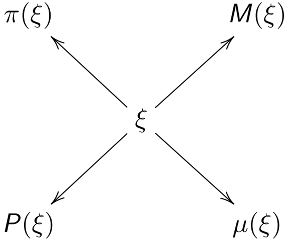

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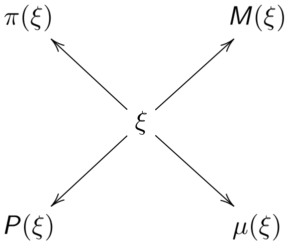

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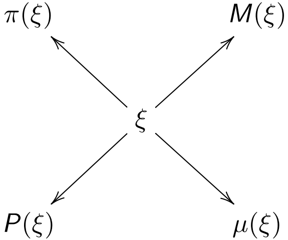

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It therefore makes sense to define **the canonical pairing**

$$\langle \cdot, \cdot \rangle_G : \, K\Pi(G/\mathbb{R}) \times KX({}^\vee G) \longrightarrow \mathbb{Z},$$

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Theorem (Adams-Barbasch-Vogan)

For $\xi,\ \xi' \in \Xi(G)$, we have

$$\left\langle \pi(\xi), P(\xi') \right\rangle_G = e(\xi)(-1)^{\dim(S_\xi)}\delta_{\xi,\xi'}.$$

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$$H = \begin{cases} \mathrm{GL}_N \\ \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_N \end{cases}$$

and let $\mathrm{Int}(s) \circ \vartheta$ be the automorphism verifying

$$^\vee G \;=\; \left((^\vee H)^{s\vartheta} \right)_0 .$$

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$$K\Pi(H(\mathbb{R},\sigma_q),\vartheta) \quad = \quad \mathbb{Z} - \text{linear space of } \left\{\mathrm{Tr}_{\vartheta}(\pi) : \pi \circ \vartheta \cong \pi\right\}$$

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We are going to define a **Twisted Pairing** :

$$\langle\cdot,\cdot\rangle_H : K\Pi(H(\mathbb{R},\sigma_q),\vartheta) \times KX({}^\vee H,\vartheta) \longrightarrow \mathbb{Z}$$

between **twisted irreducible characters** and **twisted irreducible sheaves**.

Adams-Barbasch-Vogan's approach

The automorphism $\mathrm{Int}(s) \circ \vartheta$ acts on the **ABV**-variety $X\left({}^{L}H\right)$ in a manner which is is compatible with the ${}^{\vee}H$-action.

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- It induces an action on the set $\Xi(H)$, which permits us to define

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$$\begin{aligned} KX\left({}^L H, \vartheta\right) \quad &= \quad \mathbb{Z} - \text{linear space of } \left\{\mu(\xi)^{+} : \xi \in \Xi(H)^{\vartheta}\right\} \\ &= \quad \mathbb{Z} - \text{linear space of } \left\{P(\xi)^{+} : \xi \in \Xi(H)^{\vartheta}\right\}. \end{aligned}$$

Adams-Barbasch-Vogan's approach

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The **twisted pairing** :

$$\langle \cdot, \cdot \rangle_H : K\Pi(H(\mathbb{R}, \sigma_q), \vartheta) \times KX({}^L H, \vartheta) \to \mathbb{Z}$$

may now be defined by

$$\langle M(\xi)^{\sim}, \mu(\xi')^{+} \rangle_H = \delta_{\xi, \xi'}$$

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Theorem (Adams-A-Mezo)

For all ϑ-fixed $\xi, \xi' \in \Xi(H)$, we have

$$\left\langle \pi(\xi)^{\sim}, P(\xi')^{+} \right\rangle_H = (-1)^{\dim(S_\xi)} \delta_{\xi,\xi'}.$$

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The pairing is a bridge between characters and sheaves. It allows us to regard characters and twisted characters as $\mathbb{Z}$-valued linear functionals on $KX(G)$ and $KX(H,\vartheta)$, respectively :

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$$\begin{array}{rcl} \overline{K}\Pi(G/\mathbb{R}) & \cong & \mathrm{Hom}_{\mathbb{Z}}\left(KX\left({}^{L}G\right), \mathbb{Z}\right) \\ \eta & \longmapsto & \langle \eta, \cdot \rangle_{G}, \\ \overline{K}\Pi(H(\mathbb{R}), \sigma_q) & \cong & \mathrm{Hom}_{\mathbb{Z}}\left(KX\left({}^{L}H, \vartheta\right), \mathbb{Z}\right) \\ \eta & \longmapsto & \langle \eta, \cdot \rangle_{H}. \end{array}$$

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Through this last result, we can implement a **sheaf-theoretic** version of the **(twisted) transfer map** :

$$\mathrm{Trans}_{G}^{H\rtimes\theta}:\ K\Pi(G(\mathbb{R},\sigma_q))^{\mathrm{st}} \rightarrow K\Pi(H(\mathbb{R},\sigma_q),\vartheta),$$

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  $$P(\xi),\ \xi \in \Xi(G) \longmapsto \underbrace{Q^{\mathrm{mic}}(P(\xi))}_{\text{Local system on a conormal bundle of } X(G)}$$

  such that

  $$\chi^{\mathrm{mic}}_{S_{\psi_G}}(P(\xi)) = \dim\left(Q^{\mathrm{mic}}(P(\xi))_v\right), \quad v \in T^*_S X(G).$$

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The answer lies in adapting the framework of **ABV** to Arthur's definition. We saw that Arthur defines the stable virtual character $\eta^{\mathrm{Ar}}_{\psi_G}$ as the solution of the **twisted spectral transfer identity** :

$$\mathrm{Tr}_{\vartheta}(\pi^{\sim}_{\psi}) \;=\; \mathrm{Trans}_{G}^{H\rtimes\vartheta}(\eta^{\mathrm{Ar}}_{\psi_G}(\sigma_q)).$$

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The equality between

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Let ψ_G be an A-parameter for G and write $\psi = \epsilon \circ \psi_G$ for the corresponding A-parameter of H. Then $\Pi_\psi^{\mathrm{ABV}}(\sigma_q)$ is a singleton, i.e.

$$\Pi_\psi^{\mathrm{ABV}}(\sigma_q) \;=\; \Pi_{\varphi_\psi}(\sigma_q) \;=\; \{\pi_\psi\} \;=\; \Pi_\psi^{\mathrm{Ar}}(\sigma_q).$$

Moreover

$$\begin{aligned}\mathrm{Lift}_G^{H\rtimes\vartheta}\left(\eta_{\psi_G}^{\mathrm{ABV}}(\sigma_q)\right) &= (-1)^{\dim S_\psi - \dim S_{\psi_G}}\eta_\psi^{\mathrm{ABV},+}(\sigma_q) \\ &= \mathrm{Tr}_\vartheta(\pi_\psi^\sim).\end{aligned}$$

where the $+$ symbol means that the representations occurring in $\eta_\psi^{\mathrm{ABV},+}(\sigma_q)$ are normalised through the ***Atlas normalisation****.*

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Let us mention that Atlas has already implemented a tool that compute **"weak" unipotent A-packets**, and lower rank unipotent A-packets (but the implemented tool is too slow to treat groups of higher rank).

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1. Given an irreducible representation of a classical or unitary group, implement in Atlas an algorithm that determines whether the representation belongs to an A-packet or not, and
2. if there exists such an A-packet, determine all the A-packets containing the given representation.
3. When do two A-packets have non-trivial intersection? When does an A-packet contain a tempered representation?

Structure of real A-packets : Using the microlocal definition of the map

$$\pi \longmapsto \tau_{\psi_G}^{\mathrm{ABV}}(\pi),$$

give a detailed description of its properties :

1. Under which conditions this map is injectivity/surjectivity or it lacks any of those properties.
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We expect to obtain : a more easy to handle definition, which is possible to compute in non trivial examples.

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- **Non linear real groups** ?

Thank you.