On the Discrete Burgers Equation

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Outline

Burgers' family

$$u_t + 2uu_x - \nu u_{xx} = 0 \qquad \text{(Burgers)}$$

$$u_{n,x} = u_n(u_{n+1} - u_n) \qquad \text{(sd-Burgers)}$$

$$u_{n+1,m+1} = \frac{pu_{n,m+1} - qu_{n+1,m}}{p - q + u_{n,m+1} - u_{n+1,m}} \qquad \text{(d-Burgers-I)}$$

$$\Delta_m u = \frac{(1-pu)\Delta_n(\Delta_n u + uE_n u)}{1 + q(\Delta_n u + uE_n u)}. \qquad \text{(d-Burgers-II)}$$

1. Brief review Burgers' family
2. Connection with (1+1)-D and (2+1)-D systems
3. Discrete Burgers
   - 3D consistency
   - Linearisation
   - Continuum limit

I Burgers

I-1: Burgers equation $u_t + 2uu_x - \nu u_{xx} = 0$

- [H. Bateman-1915] Motion of a viscous fluid,
- [J.M. Burgers-1948] Turbulence in a finite interval $[0, b]$:

$$bU_t = P - \frac{\nu U}{b} - \frac{1}{b}\int_0^b u^2\,\mathrm{d}x,$$
$$u_t = \frac{U}{b}u - 2uu_x + \nu u_{xx},$$

  where $u = u(x,t)$ describes velocity of the fluid, $U = U(t)$ denotes a mean velocity, P is the analogue of an exterior force acting upon U, and the terms with ν represent frictional effects.

- [V.A. Florin-1948] [E. Hopf-1950] [J.D. Cole-1951] Linear form:

$$u = -\nu\frac{\varphi_x}{\varphi}, \quad \varphi_t = \nu\varphi_{xx} + c(t)\varphi.$$

I-2: Integrability $u_t + 2uu_x - \nu u_{xx} = 0$

- Lax pair:

$$\psi_x + u\psi = \lambda\psi, \quad \psi_t = \psi_{xx} + 2u\psi_x.$$

- Burgers hierarchy:

$$u_{t_n} = -T^n u_x = -\partial_x(\partial_x + u)^n u, \quad (n = 0, 1, 2, \cdots),$$

recursion operator [P.J. Olver-1977]

$$T = \partial_x + u + u_x\partial_x^{-1} = \partial_x(\partial_x + u)\partial_x^{-1}.$$

- Linear form

$$u = -\nu\frac{\varphi_x}{\varphi}, \quad \varphi_{t_n} = \partial_x^{n+1}\varphi + c(t)\varphi,$$

- Infinitely many symmetries, conservation laws, not Hamiltonian
- Related to Bell's polynomials: $e^{y(x)}\partial_x^n e^{-y(x)}$
- Galilean trans for Burgers, Generalized GT for Burgers hierarchy

I-3: Related to (1+1)-D: derivative NLS

- Chen-Lee-Liu: [M. Wadati, K. Sogo-1983]

$$\Psi_x = M\Psi, \quad M = \begin{pmatrix} -\frac{1}{2}(\eta^2 - qr) & \eta q \\ \eta r & \frac{1}{2}(\eta^2 - qr) \end{pmatrix}, \qquad \text{(CLL sp)}$$

CLL hierarchy

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = R^m \begin{pmatrix} q \\ -r \end{pmatrix}, \quad m = 1, 2, \cdots,$$

$$R = \begin{pmatrix} \partial_x + qr + q_x\partial_x^{-1}r - q\partial_x^{-1}r_x & q_x\partial_x^{-1}q + q\partial_x^{-1}q_x \\ r_x\partial_x^{-1}r + r\partial_x^{-1}r_x & -\partial_x + qr + r_x\partial_x^{-1}q - r\partial_x^{-1}q_x \end{pmatrix}.$$

- Reduction

$$q_{t_2} = q_{xx} + 2qrq_x,$$
$$r_{t_2} = -r_{xx} + 2qrr_x,$$

derivative NLS: $r = \pm q^*$,
Burgers equation/hierarchy: $(q, r) = (u, 1)$ or $(q, r) = (1, u)$.

I-4: Related to (2+1)-D: mKP

Way-I: reduction

- Burgers: Lax pair

$$\psi_x + u\psi = \lambda\psi, \quad \psi_t = \psi_{xx} + 2u\psi_x.$$

- mKP: Lax pair

$$L_{\mathrm{mKP}}\psi = \lambda\psi, \quad \psi_{t_m} = A_m\psi,$$

where

$$L_{\mathrm{mKP}} = \partial_x + u + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots .$$

$A_m = (L_{\mathrm{mKP}}^m)_{\geq 1}$.

- Reduction

$$L_{\mathrm{mKP}}|_{u_2=u_3=\cdots=0}$$

I-4: Related to (2+1)-D: mKP

Way-II: Squared eigenfunction symmetry constraint

- mKP: Lax pair/adjoint form

$$L_{\mathrm{mKP}}\psi = \lambda\psi, \quad \psi_{t_m} = A_m\psi,$$
$$L^*_{\mathrm{mKP}}\phi = \lambda\phi, \quad \phi_{t_m} = -A^*_m\phi,$$

where

$$L_{\mathrm{mKP}} = \partial_x + u + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots .$$

- Squared eigenfunction symmetry: $\sigma = (\psi\phi)_x$
- Constraint: $0 = u_x - \sigma \implies u = \phi\psi = qr$
- Results: [D.Y. Chen-2002]

$$L_{\mathrm{mKP}} = \partial_x + q\partial_x^{-1}r\partial_x,$$

$L_{\mathrm{mKP}}\Phi = \lambda\Phi \Longrightarrow$ CLL spectral problem ($\lambda = -\eta^2$).
$\psi_{t_m} = A_m\psi,\ \phi_{t_m} = -A^*_m\phi \Longrightarrow$ CLL hierarchy.

II Semi-discrete (sd) Burgers

II-1: sdBurgers: $z_{n,t} = z_n(z_{n+1} - z_n)$ or $z_{n,t} = -z_n(z_n - z_{n-1})$

- Spectral problem: $\varphi_{n+1} = \zeta z_n \varphi_n$
- sd Burgers hierarchy: $T_1 = -\Delta E^{-1} z_n \Delta^{-1}$

$$(\ln z_n)_{t_{s+1}} = T_1 (\ln z_n)_{t_s}, \quad (\ln z_n)_{t_1} = (\ln z_n)_x = -\Delta z_{n-1} = W_1.$$

- Linear form:

$$z_n = \frac{\alpha_n}{\alpha_{n+1}}, \quad \alpha_{n,t_s} = (-1)^{s+1}\alpha_{n-s} + c_s(t)\alpha_n.$$

- Continuum limit: $z_n = e^{\varepsilon u}, \quad x = \varepsilon n, \quad \partial_{t_s} = -\varepsilon^s \partial_{t'_s}$

$$(\ln z_n)_{t_s} = (T_1 + 1)^{s-1} W_1, \quad \Longrightarrow \quad u_{t'_s} = \partial_x(\partial_x - u)^{s-1} u.$$

- BT: [Chen, Zhang, Zhang-2021]

$$\mathrm{BT}: \ z_{n,x} = -z_n(z_n - z_{n-1}), \quad (\ln z_n)_{t_{s+1}} = T_1 (\ln z_n)_{t_s}$$

$$\mathrm{Burgershierarchy}: \ \partial_{t_{s+1}} z_n = -\partial_x(\partial_x + z_n)\partial_x^{-1}\partial_{t_s} z_n.$$

II-2: Related to relativistic Toda (R-Toda)

- sdBurgers: $z_{n,t} = z_n(z_{n+1} - z_n)$ or $z_{n,t} = -z_n(z_n - z_{n-1})$
- R-Toda:

$$\Omega_{n+1} = \begin{pmatrix} \zeta(1+\alpha R_n) - \zeta^{-1} & \zeta Q_{n-1} \\ -\alpha & 0 \end{pmatrix} \Omega_n,$$

where $z_n = 1 + \alpha R_n,\ r_n = \alpha^2 Q_n$ and α is a constant.

$$\text{R-Toda}(+) \text{ hierarchy:} \quad \begin{pmatrix} \ln z_n \\ \ln r_n \end{pmatrix}_{t_{s+1}} = R \begin{pmatrix} \ln z_n \\ \ln r_n \end{pmatrix}_{t_s},$$

where

$$z_{n,t_1} = z_n(r_n - r_{n-1}),$$
$$r_{n,t_1} = r_n(r_{n+1} + z_{n+1} - r_{n-1} - z_n),$$

$$R = \begin{pmatrix} z_n & (r_n E - E^{-1} r_n)\Delta^{-1} \\ (E+1) z_n & (E r_n E - E^{-1} r_n)\Delta^{-1} + r_n + \Delta z_n \Delta^{-1} \end{pmatrix}.$$

- Reduction: $r_n = -z_n$ or $r_n = -z_{n+1}$

II-2: Related to sd derivative NLS

- sdBurgers: $z_{n,t} = z_n(z_{n+1} - z_n)$ or $z_{n,t} = -z_n(z_n - z_{n-1})$
- sdCLL(Chen-Lee-Liu): [Date,Jimbo,Miwa-1983]

$$\begin{pmatrix} \psi_{1,n+1} \\ \psi_{2,n+1} \end{pmatrix} = \begin{pmatrix} -h\eta^2 + 1 + ha_n b_n & ha_n \\ h\eta^2 b_n & 1 \end{pmatrix} \begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \end{pmatrix},$$

$$\text{sdCLL hierarchy:} \quad \begin{pmatrix} a_n \\ b_n \end{pmatrix}_{\bar{t}_{s+1}} = \bar{T} \begin{pmatrix} a_n \\ b_n \end{pmatrix}_{\bar{t}_s},$$

where

$$a_{n,\bar{t}_1} = h^{-1}(1 + ha_n b_n)(a_{n+1} - a_n),$$
$$b_{n,\bar{t}_1} = h^{-1}(1 + ha_n b_n)(b_n - b_{n-1}),$$

$$\bar{T} = \cdots$$

- Reduction: $1 + a_n = z_n,\ b_n = 1$ or $a_n = 1,\ 1 + b_n = -z_n$

II-3: Related to DΔmKP via R-Toda

- The DΔmKP hierarchy:

$$L\psi = \lambda\psi, \quad \psi_x = A_1\psi, \quad \psi_{t_s} = A_s\psi,$$
$$L^*\phi = \lambda\phi, \quad \phi_x = -A_1^*\phi, \quad \phi_{t_s} = -A_s^*\phi,$$

where ($\Delta = E - 1$, $Ef_n = f_{n+1}$, $A_s = (L^s)_{\geq 1}$)

$$L = v\Delta + v_0 + v_1\Delta^{-1} + \cdots .$$

cf. $L_{\mathrm{mKP}} = \partial_x + u + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots .$

- Squared eigenfunction symmetry: $\sigma = (\psi E\Delta^{-1}\phi)_x$,

$$v = \psi E\Delta^{-1}\phi \;\longrightarrow\; v = a_n b_n, \quad a_n = \psi, \quad b_n = E\Delta^{-1}\phi.$$

- R-Toda: $L = a_n b_n\Delta + a_n\Delta^{-1}b_n\Delta, \quad z_n = -a_n b_n, \quad r_n = a_{n+1}b_n,$

$L\psi = (a_n b_n\Delta + a_n\Delta^{-1}b_n\Delta)\psi = \lambda\psi \;\longrightarrow\;$ R-Toda spectral problem,

R-Toda hierarchy

$$\psi_{t_s} = A_s\psi, \quad \phi_{t_s} = -A_s^*\phi.$$

II-4: Related to DΔmKP via sdCLL

- The DΔmKP hierarchy:

$$\bar{L}\Phi = \lambda\Phi, \quad \Phi_{\bar{x}} = \bar{A}_1\Phi, \quad \Phi_{\bar{t}_s} = \bar{A}_s\Phi,$$
$$\bar{L}^*\Phi^* = \lambda\Phi^*, \quad \Phi^*_{\bar{x}} = -\bar{A}_1^*\Phi^*, \quad \Phi^*_{\bar{t}_s} = -\bar{A}_s^*\Phi^*,$$

$$\bar{L} = h^{-1}\bar{v}\Delta + \bar{v}_0 + h\bar{v}_1\Delta^{-1} + \cdots + h^j\bar{v}_j\Delta^{-j} + \cdots, \quad \bar{v} = 1 + h\tilde{v}.$$

- Squared eigenfunction symmetry: $\sigma = h(\Phi E\Delta^{-1}\Phi^*)_{\bar{x}}$,

$$\tilde{v} = h\Phi E\Delta^{-1}\Phi^* = a_n b_n, \quad a_n = \Phi, \quad b_n = hE\Delta^{-1}\Phi^*.$$

- sdCLL: $\bar{L} = h^{-1}(1 + ha_n b_n)\Delta + a_n\Delta^{-1}b_n\Delta$

$$\bar{L}\bar{\phi} = \lambda\bar{\phi} \longrightarrow \text{sdCLL spectral problem},$$

  sdCLL hierarchy

$$\Phi_{\bar{t}_s} = \bar{A}_s\Phi, \quad \Phi^*_{\bar{t}_s} = -\bar{A}_s^*\Phi^*.$$

III Discrete Burgers

III-1: Related to Bianchi identity

- Burgers hierarchy: $\partial_{t_{s+1}} z_n = (-1)^s \partial_x (\partial_x + z_n + \lambda)^s z_{n,x}$
- Bäcklund transformation:

$$z_{n,x} = (z_n + \lambda)(z_{n-1} - z_n),$$
$$\partial_{t_{s+1}} \ln(z_n + \lambda) = (-1)^{s+1} \Delta [E^{-1}(z_n + \lambda)]^s z_{n-1}.$$

- Bianchi/superposition formula: [Levi,Ragnisco,Bruschi-1983]

$$\begin{array}{l} \widetilde{z}_x = (\widetilde{z} - p)(z - \widetilde{z}), \\ \widehat{z}_x = (\widehat{z} - q)(z - \widehat{z}), \end{array} \quad \Longrightarrow \quad \widehat{\widetilde{z}} = \frac{p\widehat{z} - q\widetilde{z}}{p - q + \widehat{z} - \widetilde{z}}.$$

- Notations:

$$z \equiv z_{n,m}, \quad \widetilde{z} \equiv z_{n+1,m}, \quad \widehat{z} \equiv z_{n,m+1}, \quad \widehat{\widetilde{z}} \equiv z_{n+1,m+1},$$

III-2: Related to Darboux transformation

- CLL:

$$\begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \end{pmatrix}_{\bar{x}} = \begin{pmatrix} -\eta^2 + a_n b_n & \eta a_n \\ \eta b_n & 0 \end{pmatrix} \begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \end{pmatrix}.$$

- Darboux transformation:

$$\begin{pmatrix} \psi_{1,n+1} \\ \psi_{2,n+1} \end{pmatrix} = \begin{pmatrix} -h\eta^2 + 1 + h a_n b_{n+1} & h a_n \\ h\eta^2 b_{n+1} & 1 \end{pmatrix} \begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \end{pmatrix},$$

- Commuting DT:

$$M(\gamma, h, a_{n,m}, b_{n+1,m}) = \begin{pmatrix} -h\gamma + 1 + h a_{n,m} b_{n+1,m} & h a_{n,m} \\ h\gamma b_{n+1,m} & 1 \end{pmatrix},$$

$$\Psi_{n+1,m} = M(\gamma, p, a_{n,m}, b_{n+1,m})\Psi_{n,m}, \quad \Psi_{n,m+1} = M(\gamma, q, a_{n,m}, b_{n,m+1})\Psi_{n,m},$$

- Discrete CLL (reduction to dBurgers: $a_{n,m} = 1$ or $b_{n,m} = 1$):

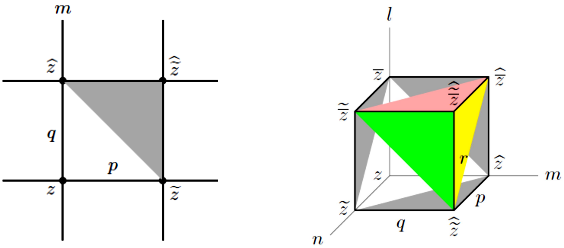
$$pa_{n,m+1} - qa_{n+1,m} - (p-q)a_{n,m} = pq(a_{n+1,m} - a_{n,m+1})a_{n,m}b_{n+1,m+1},$$
$$qb_{n,m+1} - qb_{n+1,m} + (p-q)b_{n+1,m+1} = pq(b_{n+1,m} - b_{n,m+1})a_{n,m}b_{n+1,m+1}.$$

III-3: 3D consistency of discrete Burgers

- (triangle) discrete Burgers:

$$\widehat{\widetilde{z}} = \frac{p\widehat{z} - q\widetilde{z}}{p - q + \widehat{z} - \widetilde{z}}$$

- 3D consistency:



(a) (b)

III-4: Integrability of discrete Burgers

- (triangle) discrete Burgers:

$$\widehat{\widetilde{z}} = \frac{p\widehat{z} - q\widetilde{z}}{p - q + \widehat{z} - \widetilde{z}}$$

- Lax pair:

$$\widetilde{\Phi} = U\Phi = \begin{pmatrix} p & -r\widetilde{z} \\ 1 & p - r - \widetilde{z} \end{pmatrix} \Phi, \quad \widehat{\Phi} = V\Phi = \begin{pmatrix} q & -r\widehat{z} \\ 1 & q - r - \widehat{z} \end{pmatrix} \Phi,$$

- Lax pair (without spectral parameter r):

$$\widetilde{\varphi} = (p - \widetilde{z})\varphi, \quad \widehat{\varphi} = (q - \widehat{z})\varphi.$$

- Linear form (via $z_{n,m} = p - \frac{\psi_{n-1,m}}{\psi_{n,m}}$):

$$\widehat{\psi} - \widetilde{\psi} = (p - q)\widehat{\widetilde{\psi}}.$$

Summary

Burgers' family

$$u_t + 2uu_x - \nu u_{xx} = 0 \qquad \text{(Burgers)}$$

$$u_{n,x} = u_n(u_{n+1} - u_n) \qquad \text{(sd-Burgers)}$$

$$u_{n+1,m+1} = \frac{pu_{n,m+1} - qu_{n+1,m}}{p - q + u_{n,m+1} - u_{n+1,m}} \qquad \text{(d-Burgers-I)}$$

$$\Delta_m u = \frac{(1 - pu)\Delta_n(\Delta_n u + uE_n u)}{1 + q(\Delta_n u + uE_n u)}. \qquad \text{(d-Burgers-II)}$$

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Questions for sandpit?

Thank You!

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