

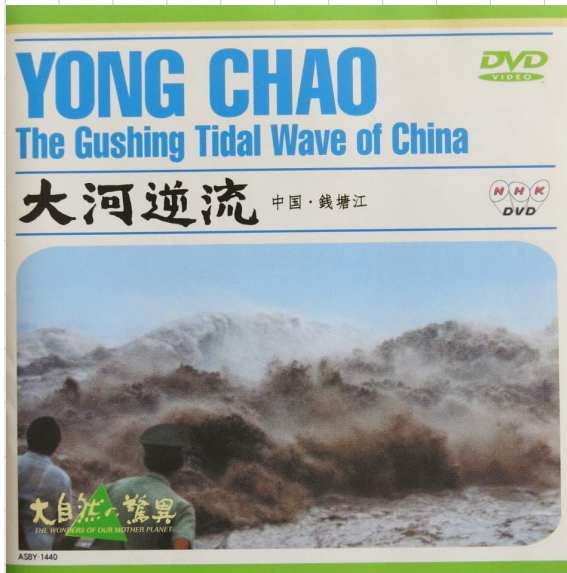
4D Wess-Zumino-Witten (WZW) models
and a unified theory of integrable systems

Masashi Hamanaka (Nagoya U.)

2023.10.24 @ 杭州

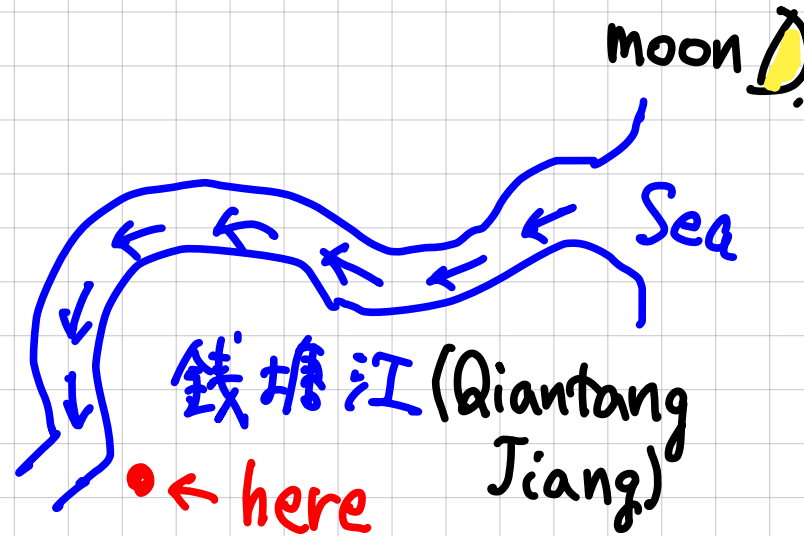
- MH, Shan-Chi Huang, Hiroaki Kanno, 2212.11800
Prog. Theor. Exp. Phys. (PTEP) 2023-4, 043B03

Hangzhou is the place of real KP solitons!

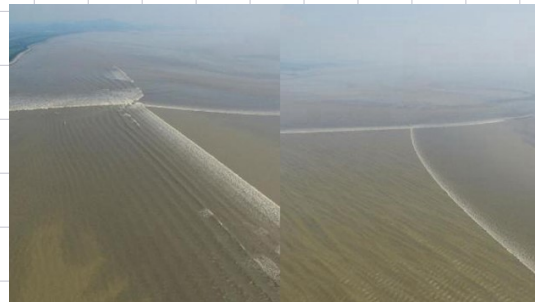


大自然の驚異 大河逆流
～中国・钱塘江～ [DVD]

(photo from Amazon)



2



1



2

1. http://japanese.china.org.cn/travel/txt/2015-08/03/content_36210294_3.htm

2. http://japanese.china.org.cn/travel/txt/2008-09/18/content_16498899.htm

§1 Introduction

②

4-dim WZW (WZW_4) model

[Donaldson '85]
[Losev-Moore-Nekrasov
-Shatashvili, '96]

• analogue of 2-dim WZW model

[Inami-Kanno-
Ueno-Xiong '96]
...

• EOM = Yang's eq \equiv Anti-Self-Dual Yang-Mills eq.
(ASD)

• In the split signature $(-,-,+,+)$,
SFT action of $N=2$ string theory

← Today we focus on
'91 [Ooguri-Vafa]

We discuss classical soliton sols. of it \rightarrow implication application

Original Motivation:

Ward's conjecture

We've made it!
(τ -fcn?)

4-dim
ASDYM

\leftrightarrow twistor theory
(so far, no Wronskian sol.)

$(-, -, +, +)$
reduction

Toda, KdV,
NLS, ...

\leftrightarrow Sato's theory
(τ -fcn \equiv Wronskian sol.)

[Ward 85, Mason-Woodhouse], ...

Original Motivation:

NC Ward's conjecture

NC = extension to 3
Noncommutative sp.

We've made it!

4-dim NC
ASDYM

↔ twistor theory

(so far, no Wronskian sol.)

(τ -fcn?)

(-,-,+,+) ↓ reduction

NC Toda, NC KdV,
NC NLS, ...

↔ Sato's theory

(τ -fcn \equiv Wronskian sol.)

NC ↔ bkg. B-field

↓ reduction

[Ward '85, Mason-Woodhouse], ...

(NC) [H-Toda '02, H'06, ...]

Quasideterminant

Reduction to KdV from ASDYM ($G = SL(2, \mathbb{C})$) 4

$$\text{ASDYM: } F_{zw} = 0, F_{z\tilde{w}} = 0, F_{z\tilde{z}} - F_{w\tilde{w}} = 0$$

① $\partial_w - \partial_{\tilde{w}} = 0, \partial_{\tilde{z}} = 0$ (dim. reduction)

② $A_{\tilde{w}} = \begin{pmatrix} 0 & 0 \\ \frac{u}{2} & 0 \end{pmatrix}, A_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_w = \begin{pmatrix} 0 & -1 \\ u & 0 \end{pmatrix}$

$$A_z = \frac{1}{4} \begin{pmatrix} u' & -2u \\ u'' + 2u^2 & -u' \end{pmatrix} \quad \begin{matrix} u = u(z, x) \\ u' = \partial_x u \end{matrix}$$

$\tilde{w} + \tilde{w}$

$$u_z - u_{xxx} - \frac{3}{2} u u_x = 0 : \text{KdV eq.} \quad \text{cf. p14}$$

\tilde{t}
 t (t, x) are real \Rightarrow not $(++++)$ but $(--++)$

What is Lagrangian for ASDYM? 📄



Candidates: (i) Yang-Mills action $\text{Tr} F_{\mu\nu} F^{\mu\nu}$

↳ ASDYM = BPS eq.

(ii) 4D WZW action

↳ ASDYM = EOM (natural?)

⇓ Ward conj

4D WZW model with the split signature $(--++)$

would be suitable? \rightsquigarrow application to $N=2$ strings

A Unified theory of integrable systems

6

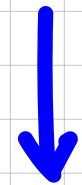
4d CS



[Costello-Yamazaki (-Witten)]

various [Delduc-Lacroix-Magro-Vicedo],
solvable models [Yoshida(K), Sakamoto,
(spin chains, PCM, ...) Fukushima, ...]

4d WZW



[Ward] [Mason-Woodhouse]

various
integrable eqs.
(KaV NLS, Toda, ...)

A Unified theory of integrable systems

6'

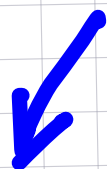
6d meromorphic

Chern-Simons (CS)

[Costello]

[Bittleston-Skinner]

4d CS



[Costello-Yamazaki (-Witten)]

various [Delduc-Lacroix-Magro-Vicedo],

solvable models [Yoshida(K), Sakamoto,

(spin chains, PCM, ...) Fukushima, ...]

...



4d WZW

[Ward]



[Mason-Woodhouse]

various
integrable eqs.
(KaV NLS, Toda, ...)

Nagoya Math-Phys Seminar Online has start!



Nagoya Math-Phys Seminar

supported by [the Ichihara International scholarship foundation](#)

2023 Autumn/2024 Winter

Date [Place]	Speaker	Title	Comment
Dec.21 (Thu) 5pm (JST) [Zoom]	Meer Ashwinkumar (Bern)	TBA (Seminar)	Zoom link will be shown here by 3pm on the seminar day.
Oct.19 (Thu) 5pm (JST) [Zoom]	Francis Howard (Benin)	Particles and p -Adic Integrals of Spin(1/2): Spin Lie Group, $R(p, q)$ -gamma and $R(p, q)$ -beta Functions, Ghost and Applications (Seminar)	Abstract , Slide , Video

2023 Spring/Summer

Date [Place]	Speaker	Title	Comment
Aug.30 (Wed)~ Sep.29 (Fri) [Zoom]	7 Plenary Speakers and 18 Parallel Speakers	Topological solitons (International Seminar-Type Online Workshop)	Workshop website
Sep.14 (Thu) 5pm (JST) [Zoom]	Frank Nijhoff (Leeds)	Lagrangian multiforms, the Darboux-KP system and Chern- Simons theory in infinite-dimensional space (Seminar)	Abstract , Slide , Video
July 21 (Fri) 9:30am (JST) [Zoom]	Atul Sharma (Harvard)	Burns holography (Seminar)	Abstract , Slide , Video
June 16 (Fri) 9:30am (JST) [Zoom]	Roland Bittleston (Perimeter)	Classical and Quantum Integrability in Self-Dual Gravity (Seminar)	Abstract , Slide , Video
June 15 (Thu) 9:30am (JST) [Zoom]	Roland Bittleston (Perimeter)	Overview of Classical and Quantum Integrability in Four Dimensions (Overview Seminar)	Abstract , Slide , Video

Plan of Talk (simple discussion)

8

§1 Introduction (15 min)

§2 Soliton Solutions of Yang's eq (7 min)

§3 4d WZW model (13 min)

§4 Conclusion & Discussion (5 min)

§4 Soliton Solutions of Yang's eq

9

Yang's eq. (on \mathbb{C}^4 : complexified space-time)

$$\partial_{\tilde{z}} \left((\partial_z \alpha) \alpha^{-1} \right) - \partial_{\tilde{w}} \left((\partial_w \alpha) \alpha^{-1} \right) = 0$$

$$\stackrel{\cong}{=} G = GL(N, \mathbb{C})$$

* Real slice

$$(z, w, \tilde{z}, \tilde{w}) \in \mathbb{C}^4,$$

$$ds^2 = dzd\tilde{z} - dwd\tilde{w}$$

$$\textcircled{1} \downarrow \begin{aligned} z &= x^1 + x^3, & w &= x^2 + x^4 \\ \tilde{z} &= x^1 - x^3, & \tilde{w} &= x^4 - x^2 \end{aligned}$$

$$\mathbb{R}^4 (+, +, -, -)$$

Ultrahyperbolic sp. \mathbb{U}

$$\textcircled{2} \downarrow \begin{aligned} z &= x^1 + ix^2, & w &= x^3 + ix^4 \\ \tilde{z} &= \bar{z}, & \tilde{w} &= -\bar{w} \end{aligned}$$

$$\mathbb{R}^4 (+, +, +, +)$$

Euclid sp. \mathbb{E}

Lax representation :

$N \times N$ const matrix



$$(*) \begin{cases} Lf = \sigma \partial_w(\sigma^{-1}f) - (\partial_{\tilde{x}} f) \tilde{\zeta} = 0 \\ Mf = \sigma \partial_z(\sigma^{-1}f) - (\partial_{\tilde{w}} f) \tilde{\zeta} = 0 \end{cases} \quad \begin{matrix} \text{right} \\ \text{action} \end{matrix}$$

compatible condition \Rightarrow Yang's eq.

$$L(M\phi) - M(L\phi) = 0$$

Darboux trf.

[Nimmo-Gilson-Okta'00] [Gilson-H-Huang-Nimmo'20]

$$(D) \begin{cases} \tilde{f} = f \tilde{\zeta} - \theta \Lambda \theta^{-1} f \\ \tilde{\sigma} = -\theta \Lambda \theta^{-1} \sigma \end{cases} \quad \begin{matrix} \theta : \text{special sol. for } \Lambda \\ N \times N \\ \text{special value} \end{matrix}$$

Under the Darboux trf. (*) is form invariant (i.e. $\tilde{L}\tilde{f} = 0$
 $\tilde{M}\tilde{f} = 0$)

n-iterations of (D) from a trivial seed sol. 11

($\sigma = 1$)

$$\sigma_n = \begin{array}{c} N \times N \\ \left| \begin{array}{cccc} \theta_1 & \dots & \theta_n & 1 \\ \theta_1^{(1)} & \dots & \theta_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-1)} & \dots & \theta_n^{(n-1)} & 0 \\ \theta_1^{(n)} & \dots & \theta_n^{(n)} & \boxed{0} \end{array} \right| \end{array}$$

$$\theta_k^{(l)} := \theta_k \Lambda_k^l$$

$$(\theta_i, \Lambda_i) : \begin{array}{l} \partial_w \theta_i = \partial_{\tilde{z}} \theta_i \Lambda_i \\ \partial_{\tilde{z}} \theta_i = \partial_{\tilde{w}} \theta_i \Lambda_i \end{array}$$

Wronskian-type!

Quasideterminant

$$\left| \begin{array}{cc} A & B \\ C & \boxed{D} \end{array} \right| := \underset{N \times N}{d - C A^{-1} B} \quad (\text{Schur complement})$$

n -soliton sols. for $G = SL(2, \mathbb{C})$:

[H-Huang, '20] 12

$$Q_n = \begin{vmatrix} \theta_1 & \dots & \theta_n & 1 \\ \theta_1^{(1)} & \dots & \theta_n^{(1)} & 0 \\ \vdots & & \vdots & \vdots \\ \theta_1^{(n-1)} & \dots & \theta_n^{(n-1)} & 0 \\ \theta_1^{(n)} & \dots & \theta_n^{(n)} & \boxed{0} \end{vmatrix}$$

$$\theta_k = \begin{pmatrix} e^{L_k} & e^{-\bar{L}_k} \\ -e^{-L_k} & e^{\bar{L}_k} \end{pmatrix}, \Lambda_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \mu_k \end{pmatrix}$$

$$L_k = \lambda_k \alpha_k z + \beta_j \tilde{z} + \lambda_j \beta_j w + \alpha_j \tilde{w}$$

(linear in space-time coord)

Rmk (U) $\mu_k = \bar{\lambda}_k, |\mu_k| = 1$
 $\Rightarrow G = SU(2)$

(E) $\mu_k = -1/\bar{\lambda}_k, |\mu_k| = 1$
 $\Rightarrow G = U(2)$

Non-abelian system

⋮

Calculate the WZW action density of them

§3 4-dim WZW model

$$\sigma(x) \in G \quad \boxed{13}$$

Action: $S_{WZW_4} = S_\sigma + S_{WZ}$

$$S_\sigma = \frac{i}{4\pi} \int_{M_4} \omega \wedge \text{Tr} \left[(\partial\sigma) \sigma^{-1} \wedge (\bar{\partial}\sigma) \sigma^{-1} \right]$$

$$S_{WZ} = -\frac{i}{12\pi} \int_{M_4} A \wedge \text{Tr} \left[(d\sigma) \sigma^{-1} \right]^3$$

$(z, w, \tilde{z}, \tilde{w})$:
local coords
of M_4

w/ $\omega = dA$: Kähler form of M_4

M_4 : flat 4-dim space-time $\omega = \frac{i}{2} (dz \wedge d\bar{z} - dw \wedge d\bar{w})$

$$d = \partial + \bar{\partial}, \quad \partial = dw \partial_w + dz \partial_z, \quad \bar{\partial} = d\bar{w} \partial_{\bar{w}} + d\bar{z} \partial_{\bar{z}}$$

EOM: $\bar{\partial} (\omega \wedge (\partial\sigma) \sigma^{-1}) = 0 \Leftrightarrow$ Yang's eq.

§3 4-dim WZW model

$$\sigma(x) \in G \quad \boxed{13}$$

Action: $S_{WZW_4} = S_\sigma + S_{WZ}$

$$S_\sigma = \frac{i}{4\pi} \int_{M_4} \omega \wedge \text{Tr} \left[(\partial\sigma) \sigma^{-1} \wedge (\tilde{\partial}\sigma) \sigma^{-1} \right]$$

$$S_{WZ} = -\frac{i}{12\pi} \int_{M_4 \times [0,1]} \omega \wedge \text{Tr} \left[(d\tilde{\sigma}) \tilde{\sigma}^{-1} \right]^3$$

$$\begin{aligned} \tilde{\sigma}(0) &= 1 \\ \tilde{\sigma}(1) &= \sigma \\ &\text{homotopy} \end{aligned}$$

w/ $\omega = dA$: Kähler form of M_4

M_4 : flat 4-dim space-time $\omega = \frac{i}{2} (dz \wedge d\bar{z} - dw \wedge d\bar{w})$

$$d = \partial + \tilde{\partial}, \quad \partial = dw \partial_w + d\bar{z} \partial_{\bar{z}}, \quad \tilde{\partial} = d\bar{w} \partial_{\bar{w}} + d\bar{z} \partial_{\bar{z}}$$

EOM: $\tilde{\partial} (\omega \wedge (\partial\sigma) \sigma^{-1}) = 0 \Leftrightarrow \text{Yang's eq.}$

N=2 string theory

14

# WS SUSY	Name	Target sp.	field contents
N=0	Bosonic String	(1+25) dim	$g_{\mu\nu}, B_{\mu\nu}, \phi, \dots$
N=1	Super string	(1+9) dim	" "
N=2	N=2 string	(2+2) dim	massless scalar only!

open N=2 string

$$\sigma = e^\varphi \leftarrow \text{the massless scalar} \quad [\text{Ooguri-Vafa, '91}]$$

$$\underbrace{\mathcal{S}_{\text{WZW}_4}}_{\text{S}_{\text{N=2 string}} \text{ (SFT)}} = \text{(in terms of } \varphi) \rightsquigarrow \text{n-pt.fcn of } \varphi$$

(coincides with WS calculations)

One soliton (on \mathbb{D})

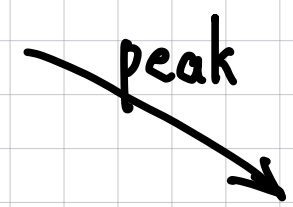
$$\sigma = -\theta \wedge \theta^{-1}, \quad \theta = \begin{pmatrix} e^L & e^{-\bar{L}} \\ -e^{-L} & e^{\bar{L}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

$$\frac{\operatorname{sech} x}{\cosh x}$$

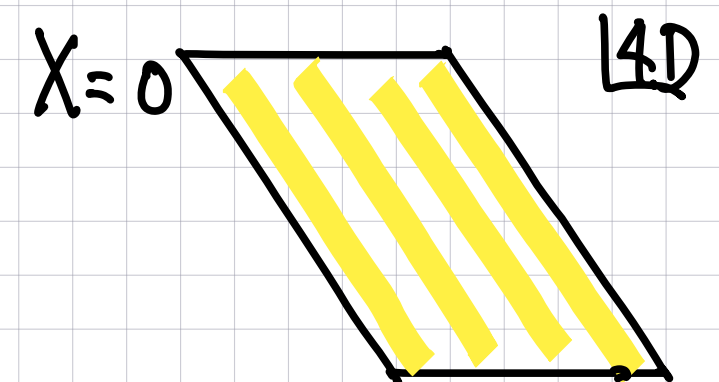


$$\mathcal{L}_\sigma = \frac{1}{8\pi} \int d_{11} \operatorname{sech}^2 X$$

$$\mathcal{L}_{WZ} \equiv 0 \text{ (identically)}$$



$$X := L + \bar{L} : \text{linear in } x^4$$



3-dim hyperplane
(codim 1)

not instanton!

One soliton (on \mathbb{D})

$\ast \lambda = \bar{\lambda} \Rightarrow \omega \equiv 0$ 15

$$\sigma = -\theta \wedge \theta^{-1}, \quad \theta = \begin{pmatrix} e^{\bar{t}} & e^{-\bar{t}} \\ -e^{-t} & e^{\bar{t}} \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

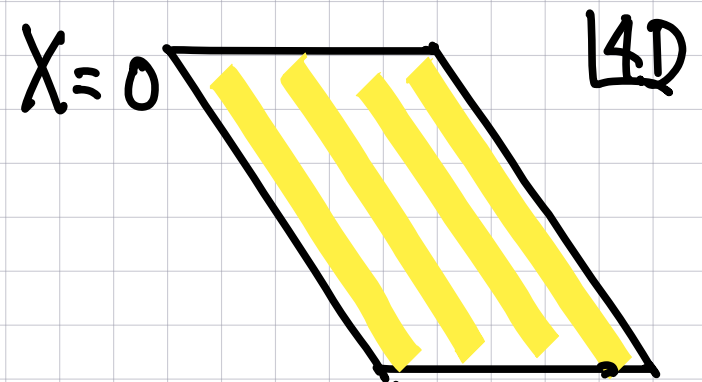
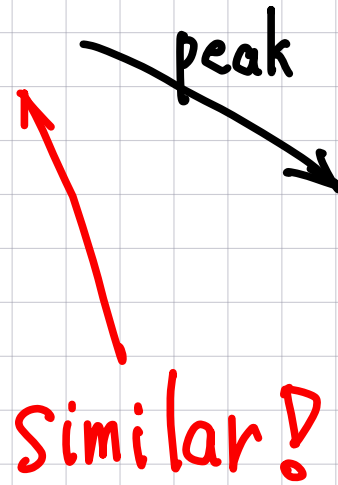
$$\begin{aligned} & \operatorname{sech} x \\ & \ll \\ & \frac{1}{\cosh x} \end{aligned}$$

$$\downarrow$$

$$\mathcal{L}_\omega = \frac{1}{8\pi} \int dt \operatorname{sech}^2 X \propto (\lambda - \bar{\lambda})^3$$

$$\mathcal{L}_{WZ} \equiv 0 \text{ (identically)}$$

$$X := L + \bar{L} : \text{linear in } x^M$$



cf. KP soliton

$$u = 2\partial_x^2 \log(e^X + e^{-X}) \propto \operatorname{sech}^2 X$$

linear in t, x, y

not instanton!

Two Soliton (\mathcal{L}_σ)

$$X_k = L_k + \bar{L}_k, \quad \Theta_{12} = \Theta_1 - \Theta_2 \quad [16]$$

$$i\Theta_k = L_k - \bar{L}_k$$

$$\mathcal{L}_\sigma = \frac{\left[A \cosh^2 X_1 + B \cosh^2 X_2 + C_\pm \cosh^2 \left(\frac{X_1 + X_2 \pm i\Theta_{12}}{2} \right) + D_\pm \cosh^2 \left(\frac{X_1 - X_2 \pm i\Theta_{12}}{2} \right) \right]}{2\pi \left(a \cosh(X_1 + X_2) + b \cosh(X_1 - X_2) + c \cos \Theta_{12} \right)^2}$$

non-singular

$$\xrightarrow{r \rightarrow \infty} \propto \operatorname{sech}^2(X_1 \pm \delta_1)$$

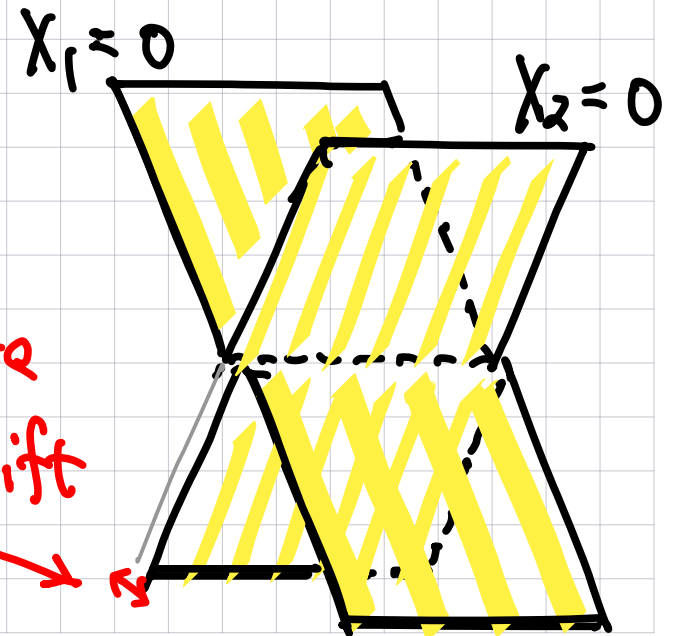
$X_1: \text{const}$

$$\xrightarrow{r \rightarrow \infty} \propto \operatorname{sech}^2(X_2 \pm \delta_2)$$

$X_2: \text{const}$

$$\xrightarrow{r \rightarrow \infty} 0$$

otherwise



phase shift
(non-linear effect)

Two Soliton (\mathcal{L}_{WZ})

17

$\mathcal{L}_{WZ} =$ (very long many terms) non-singular

$\xrightarrow{r \rightarrow \infty} 0$ (in any direction)

$\mathcal{L}_{\text{total}} = \mathcal{L}_a +$ ("dressing" in the middle region)

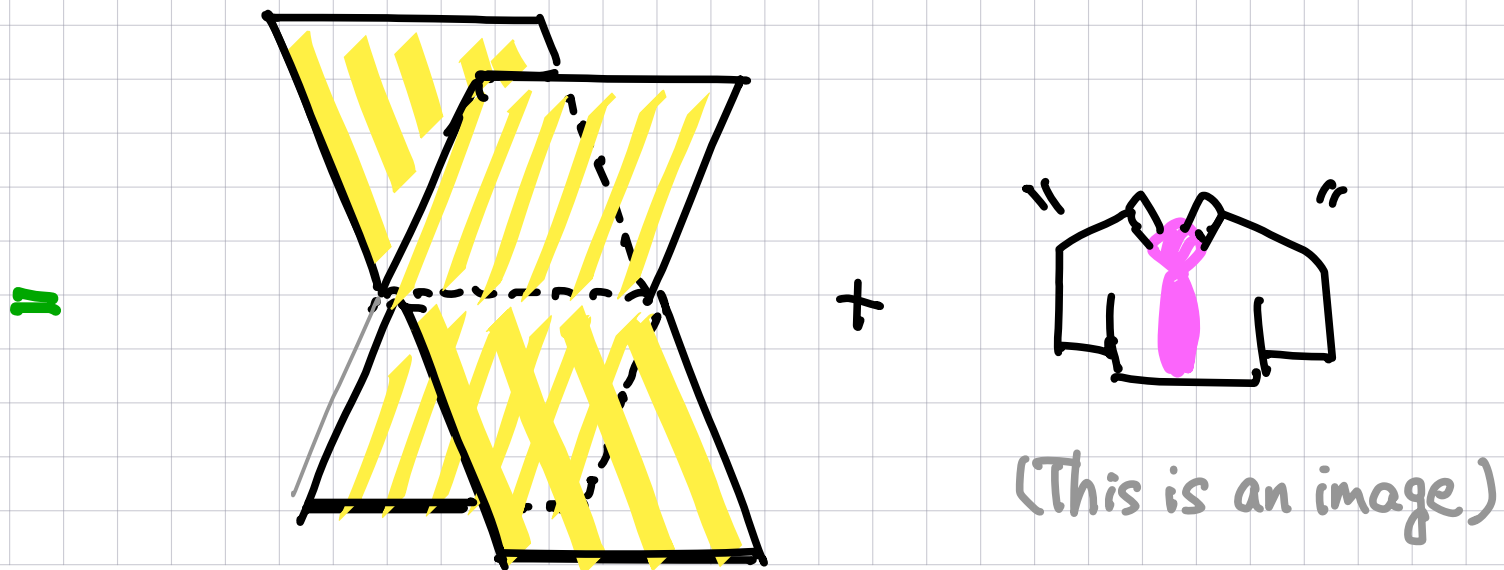
Two Soliton

17

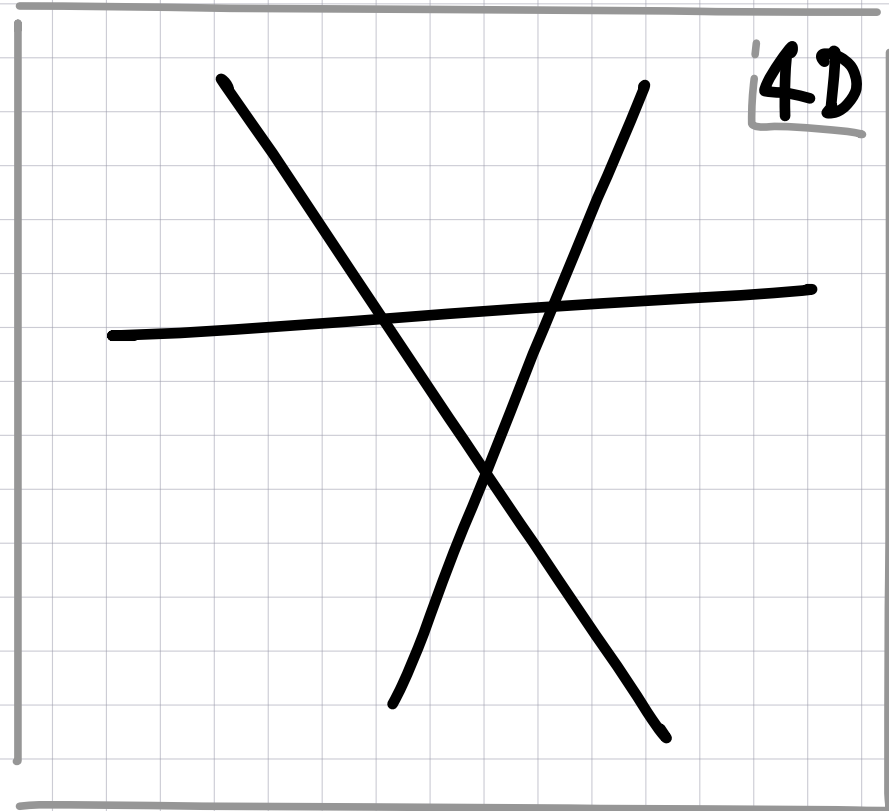
$L_{WZ} =$ (very long many terms) non-singular

$\xrightarrow{r \rightarrow \infty} 0$ (in any direction)

$L_{total} = L_a +$ ("dressing" in the middle region)



n -soliton sol. = "non-linear superposition
of n one solitons [H-Huang
'22]



intersecting n hyperplanes (with phase shifts)

Rmk 1 Reduction to (1+2) dim.

19

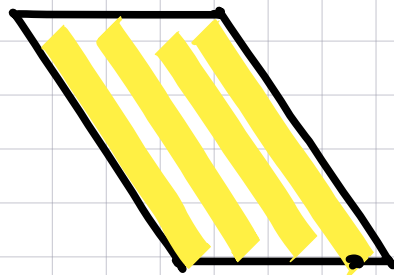
Consider $(x^1, x^2, x^3, x^4) \rightarrow (x^1, x^3, x^4)$
 $x^2 = t$ (time)

The soliton sol. $\sigma(\alpha_k = \lambda_k \beta_k)$ solves EoM in (1+2)d

⊙ $L_k = (\lambda_k \alpha_k + \beta_k) x^1 + (\lambda_k \beta_k - \alpha_k) x^2 + \dots$ ▣

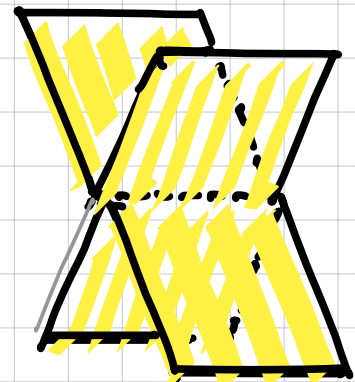
Hamiltonian $\mathcal{H} = \sum_{i=1}^3 \frac{\partial \mathcal{L}}{\partial (\partial_t \phi_i)} \partial_t \phi_i - \mathcal{L}$ ($\mathcal{H}_{wz} \equiv 0$!)

One soliton



Two soliton

(no dressing)



Energy density has the same peaks as action density.

Rmk 2 Euclidean case \mathbb{E}

20

The soliton sols. : almost the same as in \mathbb{U}

Instanton solution (well-known in YM)

(Ex) $G_{YM} = SU(2)$ 't Hooft 1-instanton

$$\mathcal{L}_a \propto \frac{(z\bar{z} + w\bar{w})^3}{\underbrace{(z\bar{z}w\bar{w})^2}_{\text{singular}} (1 + z\bar{z} + w\bar{w})^2}$$

$$\mathcal{L}_{wz} \propto \frac{(z\bar{z} + w\bar{w})(z\bar{z} - w\bar{w})^2}{\underbrace{(z\bar{z}w\bar{w})^2}_{\text{singular}} (1 + z\bar{z} + w\bar{w})^4}$$

singular

localized at the origin
(codim 4)

§4 Conclusion and Discussion

21

We constructed new-type of codim 1 solitons and calculated action densities of WZ₄ model.

↪ intersecting 3-branes in the N=2 string
(new branes)

There are many things to be seen:

- Solitonic properties (charge, mass, moduli, ...)
- Classification of the "soliton planes" cf. [Kodama-Williams '14]
- Reduced systems (YMH, Hitchin system, Ernst eq. ...)

A Unified theory of integrable systems



6d meromorphic
Chern-Simons (CS)



4d CS

← duality? →

4d WZW



various
solvable models
(spin chains, PCM, ...)

various
integrable eqs.
(KaV NLS, Toda, ...)

A Unified theory of NC integrable systems



6d meromorphic

NC Chern-Simons (CS)

NC
4d CS



← duality? →

key: Quasideterminant?

NC
4d WZW



various NC
solvable models
(spin chains, PCM, ...)

various NC
integrable eqs.
(KaV NLS, Toda, ...)

28

Seminar-Type Online Workshop on
NC Integrable Systems will be held

(probably 4 ~ 15 March 2024)

Speakers : V. Retakh, (Reviews on Q -det)
V. Roubtsov,
I. Bobrova, ...

Welcome to Join !

Thank You Very Much!

謝謝 謝謝