Homological Perspective on Splines and Finite Elements

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1 Overview

Modeling of complicated geometric objects is important in many areas of industry. For example, in the design of airplane wings, optimizing efficiency means understanding airflow, which in turn means solving a system of partial differential equations. At the opposite end of the spectrum, animation in movies such as "Toy Story" must address this same problem. *Splines* are mathematical objects which allow workers in geometric modeling and approximation theory to tackle the challenges above.

Splines are not just of practical interest in applications, but also are interesting from a purely theoretical mathematical perspective. Thus the study of splines brings together researchers from different mathematical communities: on the applied side, those working in numerical analysis and approximation theory, and on the theoretical side, those working in GKM theory, equivariant cohomology and homological algebra. One of the recent goals in the study of splines is to bring theory and practice together, and to build collaborations between researchers from different areas.

Multivariate splines, macro-elements and finite elements are established tools in numerical analysis and approximation theory. Despite decades of extensive research on the subject, there remain numerous open questions in finding their dimension, constructing local bases, and determining their approximation power. Much of what is currently known was developed by numerical analysts, using classical methods, in particular the so-called Bernstein-Bézier techniques. The 2015 MFO half workshop "Multivariate Splines and Algebraic Geometry" initiated a collaboration between numerical analysts working on splines and algebraic geometers who developed homological techniques applicable to splines. Since then there has been much progress made toward solving open problems; the research communities in geometry and topology have become involved in the intriguing structural properties of splines. The theory of Goresky-Kottwitz-MacPherson (GKM) on spaces with a nice group action shows that splines arise as equivariant cohomology rings. In terms of classical algebraic geometry, the homological tools introduced by Billera [B88] (for which he won the Fulkerson prize) lead to the study of ideals generated by powers of linear forms, then (by a classical theorem of Macaulay) to ideals of fatpoints in projective space, where there are even open questions in the projective plane. Homological algebra, in combination with algebraic topology and functional analysis, also plays a role in the numerical analysis of partial differential equations (PDEs), where it provides a rigorous foundation to the development of stable approximation methods – finite elements – that can preserve PDE-specific geometric and algebraic structures.

Our workshop brought together researchers from three different groups: numerical analysts, GKM theory and group actions, and algebraic geometers, to work on problems at the interface of the fields.

2 Scientific Progress

A key problem in both pure and applied mathematics is to construct finite dimensional spaces of functions that are capable of approximating complicated or unknown functions well. Such spaces are used in computer-aided geometric design, data fitting, and the solution of partial differential equations by the finite-element method, including the isogeometric analysis paradigm. A C^r-differentiable piecewise polynomial function on a d-dimensional simplicial complex $\Delta \subseteq \mathbb{R}^d$ is called a *spline*. Let $C^r_k(\Delta)$ denote the vector space of C^r splines on a fixed Δ, where each individual polynomial has degree at most k. There are several basic problems (finding the dimension, constructing local bases, computing their Riesz constants, and determining their approximation power), which are still open even for $d = 2$ and small values of r and k.

For general partitions, a key early result on the dimension of splines was obtained in [Schu79], where Schumaker showed that for *any* planar triangulation Δ and all k and r, the dimension of $C^r_k(\Delta)$ is bounded below by the expression

$$D^r_k := \binom{k+2}{2} + \binom{k-r+1}{2} f^0_1 - \left(\binom{k+2}{2} - \binom{r+2}{2} \right) f^0_0 + \sigma,$$

where $\sigma = \sum \sigma_i$, $\sigma_i = \sum_j \max\{(r+1+j(1-n(v_i))), 0\}$, and $n(v_i)$ is the number of distinct slopes at an interior vertex v_i. A complementary upper bound was established in [Schu84]. Using Bernstein-Bézier methods, Alfeld and Schumaker, see [AS90], [AS87], proved that the lower bound provided by D^r_k is actually equal to the dimension of $C^r_k(\Delta)$ for $k \geq 4r+1$. This was later extended to $k \geq 3r+2$ in [Hon91]. In work of [SSY] that appeared in 2020, it has been shown that the lower bound cannot hold if $k \leq 2.2r$. A central question is what happens in the range $2.2r < k < 3r+2$.

For a spline space to be useful in practice having a stable dimension that only depends on the degree (k), the order of smoothness (r), and the combinatorial properties of the partition is an essential characteristics. A stable dimension has to be complemented by optimal or sufficiently high approximation power, a property strongly related to the possibility of constructing stable bases with local support for the considered spaces [LS07]. Furthermore, the possibility of constructing the spline space locally on each of the elements of the partition is a key property for developing spline based finite elements in solving PDEs.

Although tensor-product splines enjoy the above mentioned properties, [LMS18], [MR08], they lack adequate local refinement. This triggered the interest in alternative multivariate spline structures: T-splines [LS14], hierarchical splines [GJS12] and locally refined (LR-) splines [DLP13]. All of them can be seen as special instances of splines over T-meshes [DCF06]. Unfortunately, the dimension of the spline space over a T-mesh can be unstable. Such instabilities complicate the derivation of an explicit dimension formula for any T-mesh configuration, and only lower and upper bounds can be given in the most general cases, [DCF06, M14], [TD21], [BLMRS16, BLMS19]. Could the lower and upper bounds on dimension be tightened?

Methods from homological algebra and algebraic topology, such as e.g. [MV12], have also provided new perspectives on the search for spline spaces that yield accurate numerical solutions to PDEs via the finite element method. Numerical analysts have classically relied upon functional analysis for this task but it is known that seemingly suitable spaces may fail in dangerously subtle ways, with the finite element method converging to qualitatively and quantitatively erroneous solutions. Such pitfalls can be avoided by utilizing Finite Element Exterior Calculus, [AFW06, A18], an abstract framework to guide the development of stable and accurate numerical approximations. This involves the identification of a Hilbert cochain complex associated to the given PDE and selection of finite-dimensional spline spaces that together form its cohomologically-equivalent subcomplex. The construction of such discrete complexes for splines of low regularity and tensor-product splines [BRSV11, EH13, HTHG14] is well-understood but many open questions remain for extensions to splines of high-regularity (which are particularly useful for discretizing high-order PDEs) [FN13, N15, TH21, HZZ22] and splines on unstructured and locally-refined cuboidal partitions [AH21, BSV14, ESSTV20, ST22], some of which can be related back to the dimension and completeness of spline spaces [MJG14].

Splines in representation theory and topology appear via GKM-Theory. The latter owes its name to the seminal paper [GKM98], where, among other results, Goresky, Kottwitz and MacPherson prove that, under certain technical conditions, the (equivariant) cohomology of a variety X acted upon by a torus T is encoded in a graph, namely the one skeleton of the torus action $\mathcal{G}_{(X,T)}$. After their theorem, the (equivariant) cohomology gets identified with a certain spline $C^0_\infty(\mathcal{G}_{(X,T)})$ of polynomials of unbounded degree and 0-smoothness gluing conditions. For example, one recovers the spline $C^0_\infty(AS^n)$ for the n-th Alfeld split if $X = \mathbb{P}^n_{\mathbb{C}}$ (see [Ty16]). This observation led to the definition of generalized splines on a graph [GTyV16], which deserves further investigation.

The GKM description of (equivariant) cohomology has been fruitful in the field of Schubert Calculus, where the varieties are of representation theoretic origin (flag varieties and Schubert varieties within): in this case, the existence of a nice basis, as a free module of an appropriate polynomial ring R, of $C^0_\infty(\mathcal{G}_{(X,T)})$ provides a good control on the cohomology ring. The quest for nice bases for more general varieties has interested many researchers [Go14, HTy17, LP20]. Moreover, GKM techniques led to the realisation of symmetric group representations on cohomology spaces [Ty08, LP21, CHL21]. Can some of these techniques/results be extended to $C^r_k(\mathcal{G}_{(X,T)})$? For example, the symmetric group action on $C^0_\infty(\mathcal{G}_{(X,T)})$ preserves the subspaces $C^r_k(\mathcal{G}_{(X,T)})$ and it would be certainly interesting to study such representations. Does the nice basis of $C^0_\infty(\mathcal{G}_{(X,T)})$, as a free module over R, tell us something about the dimension of $C^r_k(\mathcal{G}_{(X,T)})$ for general r?

Our workshop accelerated interactions among numerical analysts, workers in GKM theory and group actions, and algebraic geometers. The combined use of tools from the various mathematical fields could lead to essential breakthroughs on several fronts, and below we give a more detailed description of the problems covered during the workshop and areas for futher investigation.

- Behavior of the space of planar splines of polynomial degree k in the range $2.2r \leq k \leq 3r$, and study splines over T-meshes, including multidegree, variable smoothness and Tchebycheff extensions.
- Symmetric group representations on splines for the Alfeld split of arbitrary smoothness, and investigate analogous symmetries on other spline spaces.
- Upper and lower bounds for the dimension of $C^r(\Delta)$ on simplicial partitions. Especially of interest in practice is the tetrahedral ($d = 3$) case.
- Relationship between supersmoothness of multivariate splines and Macaulay inverse systems, ideals of fatpoints in projective space, and the Segre-Harbourne-Gimigliano-Hirschowitz conjecture.
- Merging Bernstein-Bézier analysis and homological techniques.
- Schubert calculus applications to determine structure, properties, and dimensions of $C^r_k(\mathcal{G}_{(X,T)})$ if X is a Schubert variety.

- Splines on polyhedral (nonsimplicial) complexes and generalized splines on arbitrary graphs.
- Stable, accurate and efficient discretizations of Hilbert complexes with a special focus on high-order PDEs, spline spaces of high (super)smoothness, locally-refined and unstructured partitions, and the existence of local commuting projections from the continuous to the discrete complex.

The workshop brought together mathematicians at diverse career stages, backgrounds, and research interests. In particular, the research talks and discussions encouraged collaboration among researchers from several distinct areas of mathematics where multivariate splines arise:

- Numerical Analysis and Approximation Theory
- Geometry and Topology, via GKM theory and equivariant cohomology
- Homological and Commutative algebra
- Numerical PDEs and Isogeometric Analysis

3 Presentations Highlights

In the first two days there were four long lectures (two each each morning) on the following topics: splines and approximation theory, finite elements, equivariant cohomology (GKM) theory, and homological algebra, to build a common vocabulary. Afternoons were largely devoted to small working groups. Mornings of days 3-5 had also shorter research talks solicited from all pre-tenure faculty and members of underrepresented groups. The scheduled talks were as follows.

Monday, May 20

- *Rebecca Goldin*: "Introduction to equivariant cohomology and GKM theory".
  Equivariant cohomology is an algebraic invariant associated to a group action on a space. GKM theory describes this graded ring in a very special set of commonly arising circumstances. In this talk, we will explain defining properties of equivariant cohomology for these spaces (with their group actions) and how to use GKM theory to calculate it. Along the way, we will explain how to associate labeled graphs, recently interpreted as splines, to these spaces.
- *Kaibo Hu*: "Finite Element Differential Complexes."
  Differential complexes are sequences of vector spaces and graded linear differential operators such that their composition vanishes. Examples of differential complexes include the de Rham complex and the Bernstein-Gelfand-Gelfand (BGG) sequences. These complexes play an important role in the analysis and numerical computation of electromagnetism and continuum mechanics. In the framework of Finite Element Exterior Calculus (FEEC), the key to obtaining stable and structure-preserving numerical methods is to use discrete (finite element, spline etc.) versions of differential complexes. The construction of such complexes has drawn increased attention and is challenging due to the tensor symmetries and continuity requirements. Finite element/spline differential complexes also provide a new perspective for some theoretical problems in spline theory, such as the dimension of spline spaces. Via exact sequences, dimension problems of a smooth scalar space can be equivalently formulated as problems of vector- or tensor-valued problems with less regularity.

  In this talk, we present some existing results and methods for constructing finite element differential complexes. This is part of an effort to vectorize or tensorize the results of scalar splines by taking differentials and completing them in a sequence.

- *Hendrik Speleers*: "Representation and Approximation of Splines: The Bernstein-Bézier Form."
  Splines are piecewise functions consisting of polynomial pieces glued together in a certain smooth way. They find application in a wide range of contexts such as computer aided geometric design, data fitting, and finite element analysis, just to mention a few. The success of univariate splines is built on two main pillars. *Representation*: spline spaces possess a special basis, called the B-spline basis, which enjoys several nice properties relevant for both theoretical and practical purposes. *Approximation*: spline spaces have an optimal approximation order for smooth functions and their derivatives. Both pillars may benefit from the local representation of the polynomial pieces in terms of Bernstein polynomials, called the Bernstein-Bézier form, which allows for a stable and geometrically meaningful description of the single pieces and their smooth connections.

  When moving to the multivariate setting, splines on triangulations emerge as a natural and powerful extension of univariate splines. Dealing with highly smooth splines on triangulations is very appealing but requires additional efforts to obtain stable dimensions and stable local bases, to achieve local constructions, and to get full approximation power. Bernstein polynomials naturally extend to triangles and the Bernstein-Bézier form generalizes in an elegant way to the triangular setting. Here the simple and clear geometric interpretation of the inter-element smoothness conditions play an even more important role in the analysis of spline spaces.

  This talk is divided into two parts. In the first part, we review the properties of univariate Bernstein polynomials and the Bernstein-Bézier form, with a focus on their geometric meaning. We also discuss their role for univariate splines, both representation and approximation. In the second part, we extend this form to splines on triangulations. Again we describe the main issues concerning representation and approximation, emphasizing the analogies with the univariate case and highlighting the difficulties arising from the multivariate setting.

- *Bert Jüttler*: "Isogeometric Analysis."
  The first part of the talk presents a brief summary of the framework of Isogeometric Analysis (IGA), which was introduced by T.J.R. Hughes et al. in 2005 as a concept to bridge the gap between design (in particular Computer Aided Geometric Design) and analysis (i.e., numerical simulation via discretization methods for PDEs such as the Finite Element Method). IGA relies on spline-based discretizations of physical fields, in particular tensor-product constructions, and methods for adaptive refinement of the resulting spline spaces are therefore of vital interest. Consequently, the second part of the talk focuses on adaptive generalizations of tensor-product splines and their mathematical properties, in particular algebraic completeness, as this is one of the main topics of this BIRS workshop. More specifically, we discuss hierarchical splines, whose origins can be traced back to the seminal work of D.R. Forsey and R.H. Bartels in 1988, splines defined by control meshes with T-joints (T-splines) as introduced by T.W. Sederberg et al. (2003), the polynomial splines over hierarchical T-meshes of J. Deng et al. (2008), and the polynomial splines over locally refined box partitions (LR-splines) of Dokken et al. (2013). Finally, the talk concludes with some suggestions for further research.

- *Beihui Yuan*: "Homological perspectives on splines."
  In this talk, I would like to draw people's attention to the dimension counting problem of spline spaces. This problem has an algebraic nature: it can be translated into the computation of Hilbert functions. Hilbert functions encode many important algebraic invariants. They are additive on short exact sequences, and that is why we can use homological methods, i.e. chain complexes, to obtain Hilbert functions of complicated objects from those of simple ones. A classical example is Billera's homological approach to prove Strang's conjecture. Later, his method was modified

by Schenck and Stillman. This set of tools has been implemented in the computer algebra system Macaulay 2.

Tuesday, May 21

- *Johnny Guzman*: "Exact finite element sequences on macro triangulations."
  Starting with C^1 spaces on macro triangulations (Alfeld, Worsey-Farin splits), we show that they are part of a finite element complex. We then discuss how some of the spaces can be used for problems in fluid flow, electro-magnetics and solid mechanics.
- *Cesare Bracco*: "Spline operators on Hilbert spaces."
  Spline functions are a well-known and widely employed tool, with applications to CAD, CAE, CAGD, numerical solution of PDEs, etc.. The theory about spline spaces has been and still is continuously growing, and led to countless types of of spline spaces: univariate and multivariate, polynomial and generalized, defined on triangulations and on (locally) tensor-product meshes, etc.. All these splines are traditionally functions $s: \mathbb{R}^m \to \mathbb{R}^n$ with varying regularities. In this presentation we explore the possibility to extend the concept of spline to operators (functions) $s: X \to \mathbb{R}$, where X is an infinite-dimensional Hilbert space, motivated by the fact that such a tool could be applied both to approximation methods and to the solution of functional differential equations. We will present a couple of constructions for piecewise k-linear operators, as a proof of concept that splines can be defined and used in this more general setting.
- *Martin Vohralik*: "Potential and flux reconstructions for optimal a priori and a posteriori error estimates."
  Given a scalar-valued discontinuous piecewise polynomial, a "potential reconstruction" is a piecewise polynomial that is trace-continuous, i.e., H^1-conforming. It is best obtained via a conforming finite element solution of local homogeneous Dirichlet problems on patches of elements sharing a vertex. Similarly, given a vector-valued discontinuous piecewise polynomial not satisfying the target divergence, a "flux reconstruction" is a piecewise polynomial that is normal-trace-continuous, i.e., H(div)-conforming, and has the target divergence. It is best obtained via local homogeneous Neumann problems on patches of elements, using the mixed finite element method. These concepts are known to lead to guaranteed, locally efficient, and polynomial-degree-robust a posteriori error estimates. We show that they also allow to devise stable local commuting projectors that lead to p-robust equivalence of global-best approximation over the whole computational domain using a conforming finite element space with local- (elementwise-)best approximations without any continuity requirement along the interfaces and without any constraint on the divergence. Therefrom, optimal hp approximation / a priori error estimates under minimal elementwise Sobolev regularity follow.
- *Thomas Grandine*: "Spline modeling with BSpy."
  Recently, Eric Brechner and I have developed an open source Python package for building spline models of points, curves, surfaces, solids, and n-dimensional manifolds called BSpy. This package, based primarily on a single object class and a handful of methods, offers a powerful capability for building and manipulating geometric models in many dimensions. This talk will explain the history and philosophy that have gone into BSpy and will demonstrate some of the surprisingly complex operations that can be performed with coding idioms that are often only a few lines of code long.

Wednesday, May 22

- *Julianna Tymoczko*: "Algebraic perspectives on splines and two tricks for computations."
  We describe in more detail a dual perspective on splines, developed by Billera and

Rose in the context of splines and by Goresky-Kottwitz-MacPherson in the context of equivariant cohomology. We then use this perspective to give two tricks that help calculations from [NST23].

- *Elizabeth Milicevic*: "Folded Alcove Walks & Applications to GKM Theory."
  This talk will explain the tool of folded alcove walks, which enjoy a wide range of applications throughout combinatorics, representation theory, number theory, and algebraic geometry. We will survey the construction of flag varieties through this lens, focusing on the problem of understanding intersections of different kinds of Schubert cells. We then highlight a key application in GKM theory.
- *Ana Maria Alonso Rodriguez*: "High Order Whitney Finite Elements: geometrical degrees of freedom."
  The talk concerns the degrees of freedom that can be used to determine univocally the fields in high order Whitney finite element spaces. There are indeed two different families of such degrees of freedom, the weights and the moments. Weights and moments coincide in the lower order case, but are rather different as soon as we consider the high order case. I will mainly focus on weights. Thank to their natural geometrical localization on the mesh, weights allow, for instance, to generalize, to the polynomial interpolation of differential k-forms, some fundamental concepts of the polynomial interpolation of regular scalar functions or to extend to the high order case graph techniques used in the low order case. I will also discuss about the relationship between weights and moments through a particular isomorphism that preserves the matrix of the gradient operator. This is a long term collaboration with Francesca Rapetti, from the University Côte d'Azur, France.
- *Jan Grošelj*: "A higher-degree super-smooth C^1 Powell-Sabin finite element."
  The Powell-Sabin 6-refinement has proven to be a convenient splitting technique for constructing smooth splines over a general triangulation. In this talk we use it to define a C^1 spline space of arbitrary degree with optimal polynomial precision and prescribed super-smoothness at split points inside triangles. Instead of traditional interpolation, we use blossoming to establish a set of functionals that characterize the spline space. The associated basis functions have some favorable properties, namely, they form a convex partition of unity and can be naturally represented in the Bernstein-Bézier form.

Thursday, May 23

- *Andrea Bressan*: "On the dimension of the space of C^{p-1} splines of degree p on the Wang-Shi split."
  The Wang-Shi split is a cross-cut partition of a triangle and consequently the dimension formula for the space of C^{p-1} splines of degree p does not have an homology term and it only involves the number of crossing lines containing each inner point. The talk will present a proof that the number of cut-lines containing any inner point is less than or equal to $p+1$ allowing to simplifying the formula to dim spline = dim polynomials + number of cut-lines.
- *Robert Piel*: "Adaptive, Structure-Preserving Finite Elements through Subdivision."
  This talk will introduce a novel construction of adaptive and yet structure preserving finite element discretizations with function spaces induced by subdivision. In many applications, for example in geophysical fluid dynamics, adaptive and structure-preserving methods can be highly beneficial to simulate the long-term evolution of a multi-scale system with several invariants of motion like the total energy. If the discretizations of such systems do not preserve these invariants, the simulation results can differ significantly from the true physical behaviour of the systems. Combining the benefits of structure preservation and adaptive finite elements is notoriously difficult. If no special care is taken, adaptive mesh refinement algorithms of standard finite element approaches usually lose the property of structure

preservation. Alternatively, the refinement can be chosen to be conforming, which in turn leads to unnecessary propagation of the refinement because surrounding cells need to be refined as well. On the other hand, IGA tensor-product techniques suffer from mesh topology restrictions. For this reason, we chose to build our function spaces upon subdivision. We extend a previous work where vector field subdivision schemes that commute with the standard vector calculus operators like the gradient or the curl were introduced. Translating their work to the finite ele- ments realm yields de-Rham-complex-preserving finite elements for scalar functions, vector fields, and density functions. We added adaptivity to their structure-preserving discretization by leveraging the hi- erarchy of the basis functions induced by the subdivision algorithm. By carefully keeping track of the introduced degrees of freedom across the refinement levels, we maintain a discrete de Rham complex and thus enable structure-preserving simulations. Our method was verified by simulating the Maxwell eigenvalue problem, a well-known test case that reproduces the analytical eigenvalues if the chosen finite element spaces constitute a discrete de Rham complex. We show that our discretization indeed yields the correct spectrum and investigate the compu-tational effort and accuracy gains of our method.

- *Alexander Woo* gave an introduction to Schubert varieties and related objects and suggested possible connections to the theme of the workshop.
- *Jeremias Arf*: "Mixed Isogeometric Methods for Hodge–Laplace Problems induced by Second-Order Hilbert Complexes."
  Through the seminal works of Buffa et al., the fruitful integration of the two discretization paradigms of Finite Element Exterior Calculus (FEEC) and Isogeometric Analysis (IGA) was demonstrated already in 2011. The latter evolved over the last nearly 20 years, stemming from the publications of Hughes et al., into a powerful concept for linking Finite Element Methods with Computer-aided design. In fact, the introduction of isogeometric discrete differential forms by Buffa et al. laid the foundation for discretizing de Rham complexes in a structure-preserving manner using B-splines. However, although the FEEC theory was derived in an abstract setting, and while Hilbert sequences play a role in various physical applications, connecting IGA and FEEC often proves challenging or is sometimes not directly clear. This is especially true for Hilbert complexes that also encompass differential operators of higher orders. We present two approaches to obtain well-posed discretizations of a whole class of Hodge–Laplace problems using IGA, while maintaining the inf-sup stability condition. We focus on mixed weak formulations of saddle-point structure and second-order Hilbert complexes. In particular, we go beyond the standard de Rham case and demonstrate that ideas from FEEC and IGA are useful for non-de Rham chains as well. A central tool for describing the underlying settings and for choosing the Finite Element spaces is the Bernstein–Gelfand–Gelfand (BGG) construction discussed by Arnold and Hu in 2021. Our approach allows us to incorporate geometries with curved boundaries, which is not directly possible with classical FEEC approaches, and also provides suitable discretizations in arbitrary dimensions. We show error estimates for both approximation methods and explain their applicability in the field of linear elasticity theory. The theoretical discussions and estimates are further illustrated with various numerical examples performed utilizing the GeoPDEs software package.

Friday, May 24

- *Ulrich Reif*: "Challenges in Isogeometric Analysis."
  The construction of spline spaces for the isogeometric analysis of higher order PDEs is partially understood for bivariate problems, but offers great challenges for trivariate problems. In this talk, we discuss the current situation and identify tasks to be addressed in the future. In particular, we consider volumetric subdivision as

a possible candidate for the construction of function spaces with sufficient Sobolev regularity. While many analytic questions still remain unsolved, we can report on progress concerning the construction of algorithms with favorable properties.

Additionally, we had two late evening sessions than we had not planned. They were the ad-hoc, on demand type of talks, and felt more like bootcamps in specific topics. They did not have specific titles or abstracts, and we list their descriptions below:

- *Johnny Guzman* led a two-hour session on exact finite element sequences for non-specialists.
- *Rebecca Goldin* and *Alex Woo* provided several examples of equivariant cohomology computations related to splines.

4 Informal Activities

The informal evening activities included brief presentations of open problems, an open forum discussion of navigating the tenure process, and grant writing. They provided the perfect venue for mentoring and interaction. The evening sessions included three workshops whose structure is described below.

1. Title: “The job hunt: postdoc/tenure-track/industry.” There is a TV ad that asks “Where do you want to go?” We started with this question, exploring various possible pathways, and then moved on to an in-depth discussion. For example, if the end goal is an academic position, a first question is “What sort of school?” Small college or PhD granting institution? If the end goal is outside academia, the question is “What type of position?” Software development, data analytics, management consulting? The discussion was led by three panelists who had taken different paths to different careers.
2. Title: “Grant writing: venues, strategies, and resilience.” Grant writing can be a daunting process. In this evening session, we discussed various aspects of grant writing, beginning with the different agencies and options, moving on to a nuts and bolts of grant writing and the utility of having a template of a successful proposal, and the importance of “thinking outside the box”. The session wrapped up with a discussion of how to deal with rejection, and ways to continue to move forward in the face of multiple rejections. One of the three panelists discussed being rejected four times in a row before succeeding.
3. Title: “Navigating the tenure process: landmarks, pitfalls, and things you might have missed.” In this session, we discussed the typical tenure model— evaluation of research, teaching, and service—as well as the important question of how different institutions weight these components. The aim was for participants to get a feeling for how pre-tenure faculty should allocate time. We also discussed the importance of having mentors, the need to ask questions, and nuts and bolts issues (such as asking for data on the most recent recipients of tenure, as well as any cases that failed). The panel was comprised of a mix of faculty who have just gone through the tenure process, and faculty who have served on P&T committees.

The absolute highlight of non-academic activities was a short presentation on Indigenous Communities given by Naim Cardinal from the “Indigenous Programs & Services” office at UBCO. It was an eye-opening experience for many participants. We highly recommend including this event into every workshop at UBCO due to both Kelowna’s and UBCO’s rich history related to Indigenous People.

The free afternoon tour was somewhat spoiled by the unpredictable weather. Most participants enjoyed a short walk in Kelowna, and visited several museums in the city.

5 Outcome of the Meeting

The last major workshop on this topic was a 2015 half workshop at MFO, organized by Tanya Sorokina, Larry Schumaker and Hal Schenck, and led to a number of collaborations. Afterwards, an INdAM workshop held in Cortona, Italy, in Septemper 2022, organized by Martina Lanini, Carla Manni and Hal Schenk offered the opportunity to consolidate and extend the collaborations. Our 2024 BIRS workshop was built on that basis and continued the conversations kickstarted at the MFO workshop and expanded at the INdAM workshop. One of the main novelties of this conference was to initiate exploration of theoretical questions that can profoundly impact the applications of multivariate splines. New groups of researchers — numerical analysts working in the field of partial differential equations (PDEs) and Isogeometric Analysis; and GKM theorists — that had been largely excluded from the 2015 MFO half workshop and marginally represented in the INdAM workshop, brought their expertise and provided novel insights on the topic.

Over the last two decades, numerical analysis has become a fertile ground for applications of multivariate splines with the introduction of isogeometric analysis. Isogeometric analysis advocates for the use of smooth splines for numerical discretization of PDEs, which would help integrate classical numerical simulations with computer-aided design and, thus, help create efficient engineering design-through-simulation workflows. This philosophy has reignited research on splines and many theoretical questions remain open. Our workshop further helped our understanding of application-oriented aspects of those theoretical questions. For instance, it helped to understand the effect of dimension instabilities on the approximation properties of spline spaces, or to study the dimension formulas in the general setting of geometrically smooth splines which find many applications in geometric modelling and numerical simulation.

Furthermore, the collaboration of numerical analysts and algebraists started during the workshop is also expected to contribute to the development of fundamentally new approaches toward numerical simulation of PDEs within the framework of finite element exterior calculus (FEEC). FEEC has shown that ideas from homological algebra, differential geometry and algebraic topology can provide a deeper understanding of the task of stable and accurate numerical discretization of PDEs. This has helped create a rigorous, abstract framework for the creation of stable and accurate numerical methods. A numerical analyst trying to use this framework will, however, need to grapple with several problems that can most naturally be addressed using tools from homological and commutative algebra. For instance, during the workshop we discovered the lack in understanding of the cohomologies of spline complexes built from locally-refinable/multiresolution spline spaces. Similarly, while the classical FEEC approach has been widely and successfully utilised for discretizing the de Rham complex, its combination with smooth splines can potentially allow us to discretise other Hilbert complexes (e.g., the elasticity complex), thus impacting simulations of many physical phenomena (e.g., elasticity, thin shell mechanics, general relativity).

Finally, the connections of GKM theory to splines have become apparent and well understood. Our workshop helped further discuss and explore their implications in the interdisciplinary setting.

In sum, our BIRS workshop has initiated strong collaborations between these communities, and we strongly believe that the combinations of methods from these areas will enable us to achieve results that are unattainable using perspective from only one of the areas.

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