BIRS 23w5006: "Spinorial and Octonionic Aspects of G_2 and $\mathrm{Spin}(7)$ Geometry" Final Workshop Report

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1 Overview of the Field

The *holonomy* of an oriented Riemannian manifold (M, g) of dimension n is a compact Lie subgroup of $\mathrm{SO}(n)$, which is a global invariant that is intimately related to the Riemann curvature tensor of g, via the Ambrose-Singer theorem. More precisely, its Lie algebra is generated by the Riemann curvature tensor of the metric. Because of this, metrics with reduced holonomy (a proper subgroup of $\mathrm{SO}(n)$) have restrictions on their curvature, which makes them interesting solutions to certain prescribed curvature equations. Note that the holonomy condition is actually a *first order* condition on the metric, which automatically implies a second order condition (a curvature constraint). In 1955, Marcel Berger classified the possible Riemannian holonomy groups that can occur. In the case that M is not locally reducible and not locally symmetric, he found that only seven possible holonomy groups could occur. These groups are summarized in Table 1.

Holonomy group	n	Name	Remarks
$\mathrm{SO}(n)$	n	oriented Riemannian	generic
$\mathrm{U}(m)$	$2m$	Kähler	complex and symplectic
$\mathrm{SU}(m)$	$2m$	Calabi-Yau	Ricci flat and Kähler
$\mathrm{Sp}(m)$	$4m$	hyperKähler	Calabi-Yau (Ricci flat) in an S^2 family of ways
$\mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$	$4m$	quaternionic-Kähler	positive Einstein (but *not* Kähler)
G_2	7	G_2 manifolds	Ricci-flat, related to octonion algebra
$\mathrm{Spin}(7)$	8	$\mathrm{Spin}(7)$ manifolds	Ricci-flat, related to octonion algebra

Table 1: The possible Riemannian holonomy groups

In particular, the last two examples are called the *exceptional* holonomy groups, as they occur in particular dimensions and are related to exceptional structures in algebra, the octonions, which was one of the themes of the workshop. It was initially thought that, although Berger could not exclude these possibilities, they would not actually occur. This was proved to not be the case, as Bryant found the first local examples in the 1980's, followed by complete non-compact examples by Bryant–Salamon and independently by groups of physicists, and later compact examples by Joyce in 1994. The last five holonomies in Table 1 (all but the generic and Kähler holonomies) are often called *special* holonomies. They are also characterized by the fact that they admit parallel or Killing spinors, which are important ingredients in theories of physics that incorporate supersymmetry. As a result, such metrics have long been of intense interest in physics. All metrics with reduced holonomy come equipped with one or more differential forms which are parallel with respect to the Levi-Civita connection.

1.1 G_2 manifolds

After Bryant first proved the local existence of metrics with holonomy G_2 in 1985, Bryant and Salamon soon constructed the first examples of complete metrics with holonomy G_2: these metrics are asymptotically conical and play a crucial role in the field. These examples justified the notion of G_2 manifold: a manifold endowed with a Riemannian metric whose holonomy is contained in G_2.

Then in 1994, Joyce constructed the first compact examples of holonomy G_2 manifolds, which was a fundamental breakthrough in the field, and the analytic theory developed by Joyce underpins all known methods to construct compact G_2 manifolds. In 2003 Kovalev gave a new construction for compact holonomy G_2 manifolds, based on a idea of Donaldson; this construction was later extended by Corti–Haskins–Nordström–Pacini. Based on these constructions, there are now known to be many examples of compact G_2 manifolds.

1.2 G_2-structures

The key to understanding and constructing G_2 manifolds goes via G_2-structures: 3-forms on 7-manifolds satisfying a certain positivity condition. A G_2-structure determines a metric and an orientation on a 7-manifold, and the condition for the G_2-structure to define a metric with holonomy contained in G_2 is the so-called *torsion-free* condition: namely that the 3-form is parallel for the Levi-Civita connection of the metric it defines or, equivalently, that it is closed and co-closed (again, using the metric and orientation that it defines). The torsion-free condition can thus be viewed as a system of (nonlinear) partial differential equations for the 3-form.

Although the main interest is in torsion-free G_2-structures, one can also consider splitting the torsion-free condition into two sub-cases: those which are closed and those which are co-closed. In fact, the co-closed condition is essentially vacuous: on any 7-manifold (compact or otherwise), a G_2-structure can be deformed to a co-closed one by the h-principle. By contrast, the closed condition is vital for all known constructions of compact G_2-manifolds, and yet is poorly understood.

1.3 Gauge theory and calibrated geometry

Donaldson–Thomas and Donaldson–Segal pioneered the notion of gauge theory in higher dimensions, and in particular in the setting of G_2 geometry. In particular, they defined G_2 instantons, which are connections generalising the more familiar anti-self-dual instantons from 4-dimensional geometry. Specifically, G_2 instantons are connections whose curvature satisfies the condition that its 2-form part lies pointwise in the Lie algebra of G_2, viewed as a subspace of the 2-forms. On G_2 manifolds, G_2 instantons are automatically Yang–Mills connections: that is, they are critical points of the Yang–Mills functional. The proposal is to try to build enumerative invariants for compact G_2 manifolds by "counting" G_2 instantons.

There is a close relationship between G_2 gauge theory and a "dual" theory of certain submanifolds. On a G_2 manifold, the G_2-structure and its Hodge dual are *calibrations*; that is, they are closed differential forms with comass one. The submanifolds calibrated by these calibrations (those submanifolds on which the forms restrict to be the volume form) are called *associative* and *coassociative* submanifolds, and they are automatically homologically volume-minimizing. There are also conjectures suggesting that one can build enumerative invariants using calibrated submanifolds.

1.4 Related geometries

There are two close cousins to G_2 geometry: $SU(3)$ geometry in 6 dimensions and $Spin(7)$ geometry in 8 dimensions.

Of particular relevance in 6 dimensions are *Calabi–Yau 3-folds* which have metrics with holonomy $SU(3)$, and *nearly Kähler* 6-manifolds which have the property that the Riemannian cone on them has a torsion-free G_2-structure. In these contexts one has associated problems in gauge theory, namely (pseudo-)Hermitian–Yang–Mills connections, and in calibrated geometry, namely (pseudo-)holomorphic curves and special Lagrangian submanifolds.

In 8 dimensions the most important geometry comes from metrics with holonomy $Spin(7)$, giving $Spin(7)$ manifolds: these include Calabi–Yau 4-folds and hyperkähler 8-folds as special cases. This yields some corresponding geometries in 7 dimensions: for example, nearly parallel G_2 manifolds, which have a co-closed G_2-structure (called nearly parallel) with the property that the Riemannian cone on them has holonomy contained in $Spin(7)$; 3-Sasakian 7-manifolds and Sasaki–Einstein 7-manifolds, where the Riemannian cone on them is hyperkähler and Calabi–Yau respectively. More generally, one can try to understand classes of $Spin(7)$-structures, which are defined by a certain very restricted type of nondegenerate 4-form on an 8-manifold. In particular, closed $Spin(7)$-structures are necessarily torsion-free and so define a metric with holonomy contained in $Spin(7)$.

1.5 Physics

Another key direction of interest in G_2 geometry comes from theoretical physics. Compact G_2 manifolds, and compact 7-manifolds with other types of G_2-structures, appear when compactifying String Theory and M-Theory, as well as in the study of anomaly cancellation in heterotic String Theory. In this context, G_2 instantons on compact G_2 manifolds are important because they minimize the Yang–Mills action, and calibrated submanifolds play a crucial role because they minimize volume.

There are several groups of researchers in theoretical physics actively pursuing G_2 geometry, and the physics perspective motivates multiple research directions in G_2 geometry for pure mathematics. In particular, the physics viewpoint leads to various predictions which remain conjectural mathematically.

Another important aspect is the role of spinors and Dirac operators in these settings. Spin geometry seems to be natural for describing many of these structures. For example, we have already mentioned that the Ricci-flat manifolds that have special holonomy admit parallel spinors. Work of Harvey and others shows that calibrations can be obtained as the square of a spinor.

2 Objectives

The first, and primary, objective of our workshop was to bring together individuals with very specific skill sets and knowledge that have not generally interacted sufficiently with each other in the past. In fact, we envisioned three such groups: (i) Riemannian geometers studying G_2 and $Spin(7)$-structures, (ii) spin geometers, and (iii) non-associative algebraists.

While octonions have been understood by algebraists for quite some time, geometers are not as familiar with all of the peculiarities of octonion algebra and its implications. An increased proficiency with octonion algebra would very likely be enormously beneficial to geometers. Therefore, our plan was to invite a few algebraists who are intimately knowledgeable about octonions in particular or non-assocative algebras in general, but know very little about their applications to Riemannian geometry. Again, such cross-fertilization promised to be extremely fruitful at least for the geometers, and hopefully to the algebraists as well.

Bringing together participants from three different research groups, especially (iii) above, also contributed to the secondary objective of our workshop, which was to increase the diversity of researchers working on exceptional Riemannian geometry, to initiate new connections and new collaborations, and to expose young researchers, who have not yet had the opportunity to sufficiently expand their mathematical breadth, to a wider and more diverse group of mathematicians. Such exposure to different viewpoints, ideas, and techniques, served to improve the abilities and achievements of all participants.

It was very important for us that the event would allow for a significant number of collaborations. To this end, we arranged the schedule so that there would be a fair number of lectures, but also a large amount of open time, with the specific goal of encouraging informal discussions during this time. Fortunately, there have been a large number of PhD students, postdoctoral researchers and other early career researchers who have joined the field in recent years. As a consequence, we deliberately made the meeting a forum for researchers at an early career stage: the vast majority of the speakers and participants were PhD students, with the rest of the speakers either postdocs or researchers who had recently been postdocs. We ran two successful Open Problem sessions where the participants described the problems and ideas in detail. More specifically, the Open Problem sessions identified several interesting research problems that the participants considered worth pursuing, which we describe in the ”Outcome of the Meeting” section.

3 Recent Developments and Open Problems

3.1 G_2 and Spin(7)-manifolds

Recently, there have been various successful generalisations of the known constructions of compact G_2-manifolds which could lead to further examples, including by Joyce–Karigiannis (who extend the Joyce construction) and Nordström (who extends the Kovalev construction). One key problem is to have similar constructions for holonomy Spin(7)-metrics.

In another direction, there has been progress in the rigorous construction of complete non-compact G_2-manifolds which had been predicted by physicists. This work by Foscolo–Haskins–Nordström produces infinitely many cohomogeneity one examples which are asymptotically conical and *asymptotically locally conical*: the latter are asymptotic to a circle bundle over a Calabi–Yau cone. Foscolo–Haskins–Nordström have also produced infinitely many asymptotically locally conical G_2-manifolds which have at most an $\mathcal{S}^1$-symmetry. In general, this is contrary to predictions from physics.

The key problem in the study of holonomy G_2 and $\mathrm{Spin}(7)-$metrics remains open:

- which compact 7-manifolds (8-manifolds) admit holonomy G_2 metrics (Spin(7)-metrics)?

We are far from having even a plausible statment of a Calabi–Yau type theorem for such manifolds, although there were many discussions about “gerbes” and their possible applications for a sufficient condition for existence of holonomy G_2-metrics.

Our understanding of this problem is incredibly limited, but there has been some progress on defining topological and analytic invariants of G_2 structures by Crowley–Goette–Nordström.

3.2 Geometric flows of G_2 and Spin(7)-structures

The question of existence of torsion-free G_2 and Spin(7)-structures on a manifold is a challenging problem. Geometric flows are a powerful tool to tackle such questions and one hopes that a suitable flow of such structures with torsion might help in proving the existence of corresponding torsion-free structures. There has been a significant amount of work in this direction, with a notable increase in activity in recent years.

The Laplacian flow for closed G_2-structures and the (modified) Laplacian co-flow for co-closed G_2-structures have been very successful with several analytic and geometric results established for them by many researchers. One particular aspect of these flows is that dimensional reductions of them in 3, 4 and 6-dimensions have been very useful for studying not only the induced flows in these lower dimensions, but they have also shed new light on the 7-dimensional situation as well. The works by Fine–Yao, Lambert–Lotay and Picard–Suan, the latter of which was also presented in the workshop, are some examples of this.

Recently, there has been a lot of activity on the harmonic flow of geometric-structures from both the general point of view in works of S Earp–Loubeau and S Earp–Fadel–Loubeau–Moreno (this was presented at the meeting) and in particular contexts of G_2, Spin(7) and Sp(2)Sp(1)-structures (this was also presented at the meeting). Apart from analytic interests in these flows themselves, one hope is that these flows can be coupled with other flows of metrics (like the Ricci flow) and the coupled flow might have "nice" properties.

Another type of geometric flow for G_2 and Spin(7)-structures which has been investigated and those which arise as the negative gradient flow of natural energy type functionals associated to these structures. We heard about the gradient flow of Spin(7)-structures in the workshop.

The key issue for the Laplacian flow of closed G_2-structures is that one needs a closed G_2-structure to start the flow and so a major open problem is:

- which compact 7-manifolds admit closed G_2-structures?

A related natural problem, which is central to the field, is:

- can a compact 7-manifold admit an *exact* G_2-structure? For example, does the 7-sphere admit a closed (and hence exact) G_2-structure?

For the harmonic flow of geometric structures, some important issues are:

- understand the singular set and singularity models for such flows, which can help us analyze the long-time behaviour of the flow;
- understand the analytic behaviour of the harmonic flow coupled with an appropriate flow of metrics.

3.3 G_2-instantons

An area where there has been a large amount of activity and recent progress is in the study and construction of G_2-instantons.

Building on the earlier gluing results of Walpuski, Sá Earp, and Sá Earp–Walpuski for G_2-instantons on the Joyce and Kovalev examples of compact G_2-manifolds, there has been a great deal of study of the relationship between G_2-instantons and associative 3-folds, and the Seiberg–Witten equations with multiple spinors on 3-manifolds. In particular, there have been significant results by Haydys, Walpuski, Haydys–Walpuski, and Doan–Walpuski.

In another direction, Oliveira, Clarke, and Lotay–Oliveira have constructed new examples and have studied the moduli space of cohomogeneity one G_2-instantons on cohomogeneity one G_2-manifolds, including the Bryant–Salamon G_2-manifolds and some examples of asymptotically locally conical G_2-manifolds. Moreover, Ball–Oliveira have constructed homogeneous G_2-instantons on Aloff–Wallach spaces (which are nearly G_2-manifolds), and have used them to distinguish between nearly parallel G_2-structures on the same Aloff–Wallach space.

In general, the key open problem in the field of G_2-instantons, aside from the many analytic issues, is:

- can G_2-instantons be used to distinguish between compact G_2-manifolds? For example, can they be so used for the known compact G_2-manifolds?

One can also ask similar questions for Spin(7)-instantons, where much less is known, and yet one expects to find many analogous results as in the G_2 setting.

An additional recent research direction has been the study of so-called deformed G_2-instantons (also known as deformed Donaldson–Thomas connections), which are "mirror" to calibrated submanifolds in a similar way to deformed Hermitian–Yang–Mills connections on Calabi–Yau manifolds. The study of these connections (and their Spin(7) analogues) is very much in its infancy, but there are many intriguing problems to explore in this setting in parallel to the more "classical" instantons.

4 Presentation Highlights

The presentations in the meeting were divided into two types of talks: 1) Longer talks of 50 minutes duration. 2) "Lightning talks" of 15 minutes duration. The latter type of talks were decided so as to have more opportunities for researchers to present work, particularly PhD students who are only partway through their studies. The lightning talks were very well received by the participants and were one of the highlights of the meeting. The research presented at the meeting can be broadly be described using 5 main interrelated themes, together with a pair of introductory talks.

- Introductory talks on octonionic and spinorial aspects of G_2 and $\mathrm{Spin}(7)$-geometry
- Instantons
- Spinors
- Special Structures
- Geometric Flows
- Calibrated Submanifolds

Many of the results discussed touched on more than one of these themes. We mention the talks below in detail with the Lightning talks marked by a $*$.

4.1 Introductory talks on octonionic and spinorial aspects of G$_2$ and Spin(7)-geometry

Speaker: **John Huerta**
Title: Octonions and spinors.
Abstract: The octonions are an eight-dimensional analogue of the complex numbers, formed by adjoining seven square roots of -1 to the real numbers, instead of just one. They are nonassociative, and thus fall outside the scope of much of the usual theory of algebras and their modules that we learn in school. Nevertheless, this strange algebra turns up in surprising corners of mathematics, essentially whenever "exceptional" structures appear. This includes the G$_2$ and Spin(7) manifolds that are our focus in this workshop. To get started with these geometric structures, I will introduce the octonions, and show how they naturally encode spinors in seven and eight dimensions.

Speaker: **Cristina Draper**
Title: The Killing's gift.
Abstract: When, in 1887, Wilhelm Killing unexpectedly found a new family of complex simple Lie groups, he gave the scientific community a precious gift: a group which can always be studied and continued to be amazing. Of course, we are talking about G$_2$, the group of the thousand facets.

4.2 Instantons

Speaker: **Daniel Platt**
Title: An example of a G$_2$-instanton on a resolution of $K3 \times T^3/\mathbb{Z}_2$ coming from a stable bundle.
Abstract: I will begin with a brief explanation of what G$_2$-instantons and G$_2$-manifolds are. There is a general construction by Joyce–Karigiannis for G$_2$-manifolds. Ignoring all analysis, I will explain one example of their construction. The example is the resolution of $K3 \times T^3/\mathbb{Z}_2$ for a very explicit $K3$ surface. Furthermore, there is a construction method for G$_2$-instantons on Joyce–Karigiannis G$_2$-manifolds. I will explain the ingredients needed for the construction, say nothing about the proof, and then explain one example of the ingredients.

Speaker: **Alfred Holmes***
Title: Spin(7) instantons and the ADHM Construction.

Abstract: In this talk I'll give an overview of a potential way to construct Spin(7) instantons from solutions to the ADHM–Seiberg–Witten equations.

Speaker: **Mario Garcia-Fernandez***
Title: Instantons from the Hull-Strominger system.
Abstract: I will explain how to construct an instanton on a real orthogonal bundle, from a solution of the Hull-Strominger system on a (possibly non-Kähler) Calabi–Yau manifold. If time allows, I will comment on how this basic principle leads to obstructions to the existence of solutions and also on conjectural extensions to the G_2 and Spin(7) heterotic systems. Based on joint work with Ral Gonzalez Molina, in arXiv:2303.05274 and arXiv:2301.08236.

Speaker: **Sergey Grigorian**
Title: Non-associative gauge theory.
Abstract: In this talk, we generalize some results from standard gauge theory to a non-associative setting. Non-associative gauge theory is based on smooth loops, which are the non-associative analogues of Lie groups. The main components of this theory include a finite-dimensional smooth loop $\mathbb{L}$, together with its tangent algebra and pseudoautomorphism group Ψ, and a smooth manifold with a principal Ψ-bundle $\mathcal{P}$. A configuration in this theory is defined as a pair (s, ω), where s is an $\mathbb{L}$-valued section and ω is a connection on $\mathcal{P}$. Each such pair determines the torsion, which is a key object in the theory. Given a fixed connection, we prove existence of configurations with divergence-free torsion, given a sufficiently small torsion in a Sobolev norm. We will also show how these results apply to G_2-geometry on 7-dimensional manifolds.

Speaker: **Izar Alonso Lorenzo**
Title: New examples of $SU(2)^2$-invariant G_2-instantons.
Abstract: G_2-instantons are a special kind of connections on a Riemannian 7-manifold, analogues of anti-self-dual connections in 4 dimensions. I will start this talk by giving an overview of why we are interested in them and known examples. Then, I will explain how we construct G_2-instantons in $SU(2)^2$-invariant cohomogeneity one manifolds and give new examples of G_2-instantons on $\mathbb{R}^4 \times \mathbb{S}^3$ and $\mathbb{S}^4 \times \mathbb{S}^3$. I will then discuss the bubbling behaviour of sequences of G_2-instantons found.

Speaker: **Leander Stecker**
Title: Reducible G_2-structures and solutions to the heterotic G_2 system.
Abstract: We discuss reducible G_2-structures, more precisely G-structures with $G \subsetneq G_2$ admitting a characteristic connection with parallel skew-torsion. We investigate how these structures can simplify the so-called heterotic G_2 system. Our study focuses on a 1-parameter deformation of the characteristic connection. We find this family to contain two G_2-instantons on 3-(α, δ)-Sasaki manifolds and a new solution of the heterotic G_2 system for arbitrary string parameter α' in the degenerate case. For further geometries we obtain approximate solutions. Joint work with Mateo Galdeano.

4.3 Spinors

Speaker: **Diego Artacho***
Title: Generalised Spin^r Structures on Homogeneous Spaces.
Abstract: Spinorial methods have proven to be a powerful tool to study geometric properties of Spin manifolds. The idea is to make accessible the power of Spin geometry to manifolds which are not necessarily Spin. The concept of Spin^c and Spin^h structures provide examples of work in that direction. In this talk, I will present a generalisation of these structures and comment on what these structures look like on homogeneous spaces, particularly on spheres.

Speaker: **Guilia Dileo***
Title: Generalized Killing spinors on 3-(α, δ)-Sasaki manifolds.
Abstract: 3-(α, δ)-Sasaki manifolds are a special class of Riemannian manifolds generalizing 3-Sasaki manifolds, and admitting a canonical metric connection with totally skew-symmetric torsion. In the

present talk I will show that every 7-dimensional 3-(α,δ)-Sasaki manifold admits a canonical G_2-structure, which determines four generalized Killing spinors. The corresponding generalized Killing numbers are explicitly obtained, providing characterization of the cases where they coincide. This is part of a joint work with Ilka Agricola.

Speaker: **Markus Upmeier***
Title: Spinors, calibrated submanifolds, and instantons.
Abstract: In the context of enumerative geometry for manifolds of special holonomy there is a deep connection between calibrated submanifolds and instantons through 'bubbling'. During the talk, I will use spinors and Dirac operators to discuss an interesting link between the (linearized) deformation theories of calibrated submanifolds and instantons. The applications of the main result include a solution to the open problem of constructing orientation data in DT-theory for Calabi–Yau 4-folds.

Speaker: **Michael Albanese**
Title: Spin^h and further generalisations of spin.
Abstract: The question of which manifolds are spin or spin^c has a simple and complete answer. In this talk we address the same question for the lesser known spin^h manifolds which have appeared in geometry and physics in recent decades. We determine the first obstruction to being spin^h and use this to provide an example of an orientable manifold which is not spin^h. The existence of such an example leads us to consider an infinite sequence of generalised spin structures. In doing so, we determine an answer to the following question: is there an integer k such that every manifold embeds in a spin manifold with codimension at most k? This is joint work with Aleksandar Milivojevic.

4.4 Special Structures

Speaker: **Henrik Naujoks***
Title: Geometry and Spectral Properties of Aloff–Wallach Manifolds (Part I).
Abstract: The focus of our attention will be the Aloff–Wallach manifolds $\mathrm{SU}(3)/{S^1}_{k,l}$. The family of manifolds depending on the embedding parameters k, l will each be equipped with a metric depending on four additional parameters. These six parameters in total lead to various interesting structures (K-contact as well as Sasakian structures, Einstein metrics, etc.) on this set of Riemannian manifolds. The interplay of these structures will be discussed. Furthermore, we investigate the spectrum of the Laplace operator: The metrics on the Aloff–Wallach manifolds $\mathrm{SU}(3)/{S^1}_{k,l}$ are not normal, but for $k = l = 1$ some of them are isometric to a normal homogeneous space. For the latter, the spectrum of the Laplace operator can be explicitly computed using methods of representation theory.

Speaker: **Jonas Henkel***
Title: Geometry and Spectral Properties of Aloff–Wallach Manifolds (Part II).
Abstract: The focus of our attention will be the Aloff–Wallach manifolds $\mathrm{SU}(3)/{S^1}_{k,l}$. The family of manifolds depending on the embedding parameters k, l will each be equipped with a metric depending on four additional parameters. These six parameters in total lead to various interesting structures (K-contact as well as Sasakian structures, Einstein metrics, etc.) on this set of Riemannian manifolds. The interplay of these structures will be discussed. Furthermore, we investigate the spectrum of the Laplace operator: The metrics on the Aloff–Wallach manifolds $\mathrm{SU}(3)/{S^1}_{k,l}$ are not normal, but for $k = l = 1$ some of them are isometric to a normal homogeneous space. For the latter, the spectrum of the Laplace operator can be explicitly computed using methods of representation theory.

Speaker: **Christina Tonnesen-Friedman***
Title: Sasakian geometry on certain fiber joins.
Abstract: This presentation will be based primarily on past and ongoing work with Charles P. Boyer. We will discuss the Sasakian geometry of certain 7-manifolds constructed by the so-called fiber join construction for K-contact manifolds, introduced by T. Yamazaki around the turn of the century. This construction can be adapted to the Sasaki case and produces some interesting examples. We will talk about some of these examples and also discuss some limitations of the construction.

Speaker: **Lucia Martin-Merchan**
Title: Topological properties of closed G_2 manifolds through compact quotients of Lie groups.
Abstract: In this talk, we discuss two problems where compact quotients of Lie groups are useful for understanding topological properties of compact closed G_2 manifolds that don't admit any torsion-free G_2 structure. These problems are related to the questions: Are simply connected compact closed G_2 manifolds formal? Could a compact closed G_2 manifold have third Betti number $b_3 = 0$?

Using compact quotients of Lie groups, we first outline the construction of a manifold admitting a closed G_2 structure that is not formal and has first Betti number $b_1 = 1$. Later, we show that compact quotients of Lie groups do not have any invariant G_2 structure. The last result is joint work with Anna Fino and Alberto Raffero.

Speaker: **Anton Iliashenko**
Title: Betti numbers of nearly G_2 and nearly Khler manifolds with Weyl curvature bounds.
Abstract: We use the Weitzenbck formulas to get information about the Betti numbers of nearly G_2 and nearly Khler manifolds. First, we establish estimates on two curvature-type self adjoint operators on particular spaces assuming bounds on the sectional curvature. Then using the Weitzenbck formulas on harmonic forms, we get results of the form: if certain lower bounds hold for these curvature operators then certain Betti numbers are zero. Finally, we combine both steps above to get sufficient conditions of vanishing of certain Betti numbers based on the bounds on the sectional curvature.

Speaker: **Gavin Ball**
Title: Irreducible SO(3)-geometry in dimension 5.
Abstract: The action of $\mathrm{SO}(3)$ by conjugation on the space of symmetric traceless matrices gives an embedding of $\mathrm{SO}(3)$ in $\mathrm{SO}(5)$. A 5-manifold whose structure group reduces to this copy of $\mathrm{SO}(3)$ is said to carry an $\mathrm{SO}(3)$-structure. The integrable examples of these structures are the symmetric spaces $\mathbb{R}^5$, $\mathrm{SU}(3)/\mathrm{SO}(3)$ and $\mathrm{SL}(3)/\mathrm{SO}(3)$, and general $\mathrm{SO}(3)$-structures may be thought of as non-integrable analogues of these spaces. In my talk, I will describe work in progress on the local geometry of a subclass of $\mathrm{SO}(3)$-structures called the nearly integrable $\mathrm{SO}(3)$-structures. The nearly integrable condition was introduced by Bobienski and Nurowski as an analogue of the nearly Khler condition in almost Hermitian geometry. However, despite the similarity of the definitions, it turns out that the local geometry of nearly integrable $\mathrm{SO}(3)$-structures is significantly more restricted compared to the nearly Khler case. The rigid nature of the local geometry suggests the possibility of giving a global classification of nearly integrable $\mathrm{SO}(3)$-structures and I will sketch out such a program. If time permits, I will describe relations with G_2-geometry.

Speaker: **Fabian Lehmann**
Title: Closed 3-forms in dimension 5.
Abstract: There is a notion of non-degenerate 3-form in six and seven dimensions which are the pointwise model for G_2- and $\mathrm{SL}(3,\mathbb{C})$-structures, respectively. These are directly related, as the restriction of a 3-form which defines a G_2-structure on a 7-manifold to a real hypersurface induces an $\mathrm{SL}(3,\mathbb{C})$-structure. I will describe the geometric structure induced on a real hypersurface inside a 6-manifold with an $\mathrm{SL}(3,\mathbb{C})$-structure under a certain convexity condition. This is based on joint work with S. Donaldson.

4.5 Geometric Flows

Speaker: **Udhav Fowdar**
Title: On the harmonic flow of Sp(2)Sp(1)-structures on 8-manifolds.
Abstract: The harmonic flow of an H-structure (aka the isometric flow) is the gradient flow of the energy functional for the intrinsic torsion. In recent years the cases when $H = \mathrm{U(n)}, \mathrm{G}_2$ and Spin(7) have been studied in great detail mainly due to their relation with special holonomy. In this talk I will discuss the case when $H =$ Sp(2)Sp(1) (i.e. the quaternionic Khler case) and shed some light into how the representation theory of H allows for a more unified approach. I will also discuss the cases when $H =$ Sp(1) and Sp(2) to illustrate certain similarities and differences. Aside from analytical

aspects of the flow, I will also describe explicit examples of non-trivial harmonic H-structures and as well as soliton solutions to the flow. This is a joint work with Henrique S Earp.

Speaker: **Gonalo Oliveira**
Title: Lagrangian mean curvature flow and the Gibbons-Hawking ansatz.
Abstract: In this talk, I will report on joint work with Jason Lotay on which we prove versions of the Thomas and Thomas-Yau conjectures regarding the existence of special Lagrangian submanifolds and the role of Lagrangian mean curvature flow as a way to find them. I will also report on some more recent work towards proving more recent conjectures due to Joyce.

Speaker: **Henrique S Earp**
Title: Flows of geometric structures.
Abstract: We develop an abstract theory of flows of geometric H-structures, i.e., flows of tensor fields defining H-reductions of the frame bundle, for a closed and connected subgroup $H \subset \mathrm{SO}(n)$, on any connected and oriented n-manifold with sufficient topology to admit such structures.

The first part of the talk sets up a unifying theoretical framework for deformations of such H-structures, by way of the natural infinitesimal action of $\mathrm{GL}(n, \mathbb{R})$ on tensors combined with various bundle decompositions induced by H-structures. We compute evolution equations for the intrinsic torsion under general flows of H-structures and, as applications, we obtain general Bianchi-type identities for H-structures, and, for closed manifolds, a general first variation formula for the L^2-Dirichlet energy functional $\mathcal{E}$ on the space of H-structures.

We then specialise the theory to the negative gradient flow of $\mathcal{E}$ over isometric H-structures, i.e., their harmonic flow. The core result is an almost monotonocity formula along the flow for a scale-invariant localised energy, similar to the classical formulae by Chen–Struwe for the harmonic map heat flow. This yields an ϵ-regularity theorem and an energy gap result for harmonic structures, as well as long-time existence for the flow under small initial energy, with respect to the L^∞-norm of initial torsion, in the spirit of Chen–Ding. Moreover, below a certain energy level, the absence of a torsion-free isometric H-structure in the initial homotopy class imposes the formation of finite-time singularities. These seemingly contrasting statements are illustrated by examples on flat n-tori, so long as $\pi_n(\mathrm{SO}(n)/H) \neq \{1\}$; e.g. when $n = 7$ and $H = \mathrm{G}_2$, or $n = 8$ and $H = \mathrm{Spin}(7)$.

Speaker: **Caleb Suan**
Title: Flows of G_2-structures associated to Calabi–Yau manifolds.
Abstract: The Laplacian flow and coflow are two of the most studied flows in G_2 geometry. We will establish a correspondence between parabolic complex Monge–Ampre equations and these flows for initial data on a torus bundle over a complex Calabi–Yau 2- or 3-fold given from a Khler metric. We will use estimates for these complex Monge–Ampre flows to show that both the Laplacian flow and coflow exist for all time and converge to a torsion-free G_2 structure induced by a Ricci-flat Khler metric. This is joint work with Sbastien Picard.

Speaker: **Shubham Dwivedi***
Title: A gradient flow of Spin(7)-structures.
Abstract: We will introduce a geometric flow of Spin(7)-structures which is the negative gradient flow of a natural energy functional on the space of Spin(7)-structures. We will evaluate the evolution of the Riemannian metric and show that the flow exists for a short time.

4.6 Calibrated geometry

Speaker: **Jesse Madnick**
Title: Harmonic Spinors and Associative 3-folds.
Abstract: There are several relationships between (twisted) harmonic spinors and associative submanifolds. For example, the Dirac operator appears in the PDE for associative graphs, in the deformation theory for associative submanifolds, and in the second variation formula for volume.

In the first part of this talk (joint work with Gavin Ball), we provide another relationship. If a 7-manifold M has a closed or nearly-parallel G_2-structure, we show that the second fundamental form of an associative can be viewed as a twisted spinor. Moreover, if M has constant curvature (e.g., if $M = \mathbb{R}^7, \mathbb{S}^7$, or $\mathbb{T}^7$), then this twisted spinor is harmonic. Intuitively, this is a spin-geometric analogue of the classical Hopf differential for 2-dimensional surfaces.

In the second part, we elaborate on this theme, highlighting the many analogies between harmonic spinors and holomorphic objects. In particular, we provide a taxonomy of Dirac equations that have arisen in the literature, which in turn suggests several avenues for further work.

Speaker: **Da Rong Cheng**
Title: A variational characterization of calibrated submanifolds.
Abstract: I will report on recent joint work with Spiro Karigiannis and Jesse Madnick where we discover, for a number of different calibrations, a characterization of calibrated submanifolds in terms of the first variation of the volume functional with respect to a special set of deformations of the ambient metric determined by the calibration form. Generalizing earlier such results due to Arezzo and Sun for complex submanifolds, we obtain variational characterizations for associative 3-folds and coassociative 4-folds in manifolds with G_2-structures, as well as for Cayley 4-folds in manifolds with Spin(7)-structures.

Speaker: **Federico Trinca**
Title: Calibrated geometry in G_2-manifolds with cohomogeneity two symmetry.
Abstract: Constructing associative and coassociative submanifolds of a G_2-manifold is, in general, a difficult task. However, when the ambient manifold admits symmetries, finding cohomogeneity one calibrated submanifolds is more tractable. In this talk, I will discuss joint work with B. Aslan regarding the geometry of such calibrated submanifolds in G_2-manifolds with a non-abelian cohomogeneity two symmetry. Afterwards, I will explain how to apply these results to describe new large families of complete associatives in the Bryant–Salamon manifold of topology $S^3 \times \mathbb{R}^4$ and in the manifolds recently constructed by Foscolo–Haskins–Nrdstrom.

5 Scientific Progress Made

We summarize the scientific progress made in each of the main themes highlighted in the previous section.

5.1 Instantons

There has clearly been a significant increase in our understanding of higher-dimensional gauge theory beyond the established settings of compact Calabi–Yau, G_2, and Spin(7)-manifolds. There have been extensions to non-integrable structures, such as almost complex 6-manifolds, nearly parallel G_2-manifolds, and Sasaki–Einstein 7-manifolds, and in the study of the non-compact setting of asymptotically conical G_2-manifolds. In particular, we have seen classification and deformation theory results. Moreover, there is interest in a non-associative gauge theory on manifolds with G_2-structure.

In the compact and non-compact G_2-manifold setting, which holds the greatest interest in G_2 geometry, there has been exciting progress towards potentially constructing a large number of G_2-instantons on the new examples of G_2-manifolds due to Joyce–Karigiannis as well as instantons on non-compact examples of G_2-manifolds due to Bryant–Salamon and Foscolo–Haskins–Nordstrm.

A new avenue has been study of the Hull–Strominger system and solutions to the heterotic G_2-system, as well as their relation to the study of moduli spaces of G_2-instantons.

5.2 Spinors

A lot of progress has been made in the study of special structures on manifolds arising from spinors. There have been spinorial classifications of manifolds with G_2 and Spin(7)-structures. There have also been many results on properties of the Dirac operator, in particular, in studying the spectrum of the Dirac operator and applications to understanding the geometric properties of manifolds. There have been many significant results on the relationship between spinors and calibrated submanifolds of manifolds with special holonomy, and spinors have also been used to study properties such as the orientation of the moduli spaces of instantons. Further generalisations of a spin structure like the Spin^r and Spin^h-structures have also been studied.

5.3 Special Structures

Special structures have received relatively little detailed attention and are generally quite poorly understood. The results presented in the meeting clearly show a marked improvement in our ability to study and understand these structures. For example, we learned about advances in our knowledge of the spectral properties of a class of nearly parallel G_2-manifolds. and saw very interesting results on topology of manifolds with closed G_2-structures and nearly parallel G_2 manifolds. We also saw the geometry of nearly integrable $\mathrm{SO}(3)$-structures in dimension 5 and their relationship to G_2-geometry. In similar themes, there were exciting results about the geometry induced by closed 3-forms in dimension 5. The techniques described in these talks will definitely have many application in related problems.

5.4 Geometric Flows

There were some interesting results concerning the dimensional reduction of the G_2-Laplacian flow and the G_2-Laplacian co-flow and its applications, building on the general theory developed in recent years. Specifically there were some impressive long-time existence and convergence results in the setting of reducing the flow to dimensions 6 and 4 and understanding the relationship with the more well-understood Khler–Ricci flow and the so called Monge–Ampre flow. Both of these results certainly merit further examination and reveal exciting future research avenues for investigation.

The analytic foundations were developed for the harmonic flow of geometric structures which has G_2, $\mathrm{Spin}(7)$ and Sp(2)Sp(1) as special cases. This is a new research topic that has links to several research groups in geometry of manifolds with special holonomy, and so will certainly continue to be studied.

A new flow of Spin(7)-structures was introduced and analytic properties of the flow were discussed. Since flows of Spin(7)-structures have not been studied in much detail before it is expected that the flow discussed during the meeting will be studied in much more detail in the future.

There were exciting results on the Lagrangian mean curvature flow and its applications to prove versions of conjectures due to Thomas, Thomas–Yau and Joyce. The discussion emphasized the strength of geometric flows techniques to prove hard conjectures on the existence of calibrated submanifolds inside Calabi–Yau manifolds.

5.5 Calibrated Submanifolds

There was notable progress made in the study of calibrated submanifolds both in special holonomy manifolds and outside of the setting of well-known areas of manifolds with special holonomy equipped with their usual calibrations. We saw a variational characterisation of calibrated submanifolds of manifolds with exceptional holonomy. Symmetry methods are a powerful tool in constructing new examples of geometric objects and we had a discussion on cohomogeneity two associatives in noncompact G_2 manifolds constructed by Bryant–Salamon and Foscolo–Haskins–Nrdstrom. There was a very nice discussion on spinors and the second fundamental form of associatives in nearly G_2 manifolds. In particular, the techniques developed in these works are certainly to yield further results in related areas.

6 Outcome of the Meeting

The key outcome of the meeting was the increase in communication and collaboration between researchers in G_2 and Spin(7)-geometry who work on seemingly different aspects of these structures, which has and will continue to lead to exciting new research directions and results. It is particularly worth emphasizing the positive outcome of the meeting for early career researchers present, mainly for PhD students but also some postdocs and other participants, who unanimously expressed how enjoyable and productive the meeting was for them. Senior researchers also remarked on how refreshing it was to have so many early career researchers interacting significantly with them and each other, which provided a unique opportunity to learn about and to offer input towards the research avenues pursued by the next generation of researchers in the field.

More specifically, the Open Problem sessions identified several interesting research problems that the participants considered worth pursuing, which we describe below.

1. **(Jason Lotay)** Recall the notion of "triality" from John Huerta's talk. We know about triality at the algebraic level: it is a symmetry between vectors and spinors for the normed division algebra $\mathbb{O}$. Suppose we have (M^8, Φ) which is an 8-dimensional manifold with a torsion-free $\mathrm{Spin}(7)$-structure Φ. Is there any geometric meaning (as opposed to the purely algebraic structure at the level of Clifford algebras and $\mathbb{O}$) of triality in this setting?

   Does it make sense to define the notion of "mirror triality" for a triple of objects similar to the notion of mirror manifolds and mirror symmetry?

   Some ideas related to the first question were suggested by **Gavin Ball:** If we look at $Gr_{\text{Cayley}}(4, 8)$, the Grassmannian of Cayley 4-planes in $\mathbb{R}^8$ and the Grassmannian of 3-planes in $\mathbb{R}^7$ then $Gr_{\text{Cayley}}(4, 8) \cong Gr(3, 7)$.

   If we look at the space of curvature tensors of a Spin(7)-manifold, i.e., (M^8, Φ) with a torsion-free $\mathrm{Spin}(7)$-structure Φ then that as an irreducible $\mathrm{Spin}(7)$-representation is isomorphic to $V_{0,2,0}$ which is also isomorphic to the space of curvature tensors of Ricci-flat 7-manifolds. This might give a hint for the geometric implication of triality for $\mathrm{Spin}(7)$-manifolds.

   In fact, an analogous question would be that if we have two $\mathrm{Spin}(7)$-manifolds M^8_+, M^8_- then do they relate to a Ricci-flat 7-manifold if we have a geometric notion of triality?

2. **(Spiro Karigiannis)** If b^2, b^3 denote the 2nd and 3rd Betti numbers of a G_2-manifold then $b^2 + b^3$ is invariant under "mirror symmetry" for G_2-manifolds, i.e, they remain the same for the mirror manifolds. There is a notion of conifold transition in Calabi–Yau geometry and an analogous idea of G_2 conifold transitions has been given by Atiyah–Witten [1]. Recall that a G_2-manifold M^7 is called **semi-flat** if M is a coassociative fibration and the fibers are flat tori T^4. What can be said about the G_2 conifold transitions in the semi-flat case?

3. **(Jesse Madnick)** Construct non-trivial compact associative submanifolds in the Aloff–Wallach spaces $N_{k,l}$ with $(k, l \neq (1, 1))$ where $N_{k,l} = \dfrac{\mathrm{SU}(3)}{\mathrm{U}(1)_{k,l}}$ with its homogeneous nearly parallel G_2-structure.

   What can be said about the conformal structure of associatives $\Sigma^3 \subset M^7$? An idea would be to use harmonic spinors just like one uses holomorphic sections to study conformal structures for holomorphic curves.

   Can an **open** Riemann surface be conformally embedded in S^6? It's a theorem due to Robert Bryant that closed Riemann surfaces can be conformally embedded in S^6.

4. **(Sergey Grigorian, Spiro Karigiannis, John Huerta.)** Consider **gerbes** on G_2-manifolds. Suppose we have a manifold with a closed G_2-structure, i.e., (M^7, φ) with $d\varphi = 0$ and $[\varphi] \in H^3(M, \mathbb{Z})$. Is there any relation between $\mathrm{U}(1)$-gerbes on (M^7, φ) and $[\varphi]$. Or, consider $d{*}\varphi = 0$ and $[*\varphi] \in H^4(M, \mathbb{Z})$. Does there exist a relation between a 2-gerbe over M^7 and $[*\varphi]$?

   A motivation to study these questions come from Kähler geometry and to try to come up with a "G_2-Calabi–Yau theorem". A more precise but still vague question is the following: Recall that if we have a Kähler manifold (M^{2m}, g, J, ω) and we consider the canonical bundle $K = \Lambda^{m,0}(T^*M)$, then it is a line bundle over M, and Yau's proof of the Calabi conjecture states that $c_1(K) = 0 \iff$ there exists a Ricci-flat metric with its Ricci form in $[\omega]$. Here $c_1(K)$ is the first Chern class of K.

So now suppose we have a manifold with a G_2-structure (M^7, φ, g). Does there exist some "canonical gerbe K" on M such that $c_1(K) = 0 \in H^3(M, \mathbb{Z}) = 0 \iff$ there exist a torsion-free G_2-structure in $[\varphi]$? Here $c_1(K)$ is the "first Chern class of the gerbe K" or more precisely the Dixmier–Douady class. Note that the question as stated is particularly vague because we still do not understand the actual notion of gerbes on G_2-manifolds and their relation to the torsion-freeness of the G_2-structure.

More information about gerbes can be found in [2, 3, 4].

5. **(Mario Garcia-Fernandez)** Consider the heterotic G_2-system: that is, we have a compact (M^7, φ), $P \to M$ is a principle G-bundle with compact G and A a connection on P, and let $\alpha' \in \mathbb{R}$ be such that $dH = \alpha' \langle F_A \wedge F_A \rangle$, where $H \in \Omega^3(M)$.

   (a) Is there a spinorial interpretation for the cases when the torsion component $\tau_0 \neq 0$?

   (b) There have been both exact and approximate solutions of the heterotic G_2-system. Construct large classes of solutions and maybe solutions with large volume?

   (c) Is there a geometric flow to study the heterotic G_2-system?

   (d) From considerations in physics, α' is hoped to be "small". Do the solutions proposed in Leander Stecker's talk in conference (based on his work with Mateo Galdeano) have small α'?

   (e) Consider a sequence of heterotic G_2-systems $\{(M_{\alpha'_n}, \varphi_{\alpha'_n}, A_{\alpha'_n})$ where α'_n is a sequence in $\mathbb{R}$ and suppose that $\alpha'_n \to 0$. What can be said about the limit?

   (f) In the case of part (e), suppose that $M_{\alpha'_n} = M_{\alpha'} = M$ and that it admits a torsion-free G_2-structure φ. Do we have $\varphi_{\alpha'_n} \to \varphi$ and $A_{\alpha'_n} \to A$ with A a G_2-instanton?

   (g) If $M_{\alpha'}$ does not have a torsion-free limit then what happens to the limit? Does the limit collapse? Is the limit a soliton? (There is a notion of a heterotic G_2-system being a soliton)

   (h) Can we construct solutions of the heterotic G_2-system with $\alpha' \neq 0$ from a limit?

6. **(Henrique S Earp)** Consider instantons of Sasakian 7-manifolds, i.e., we take $\sigma \in \Omega^3(M)$ with $\sigma = \eta \wedge d\eta$ with η the contact 1-form to define the notion of instanton. For the Sasakian case (which could be viewed as transverse-Khler geometry) we have $d\eta = \omega$ and hence $\sigma = \eta \wedge \omega$. The space of 2-forms decompose further with

$$\Omega^2 = \Omega^2_V \oplus \Omega^2_H$$

and furthermore

$$\Omega^2_H = \Omega^2_8 \oplus \Omega^2_6 \oplus \Omega^2_1.$$

Instantons A with $F_A \in \Omega^2_8$ are self-dual contact instantons. If we look at the moduli space of self-dual contact instantons $\mathcal{M}_{SDCI}$ then one can show that dim $\mathcal{M}_{SDCI} =$ ind $\not{D}$ and $\mathcal{M}_{SDCI}$ is Khler on its smooth locus.

   (a) Can we define an orientation on $\mathcal{M}_{SDCI}$?

   (b) What happens to the blow-ups, bubbling, and compactifications of $\mathcal{M}_{SDCI}$?

   (c) Suppose we consider the 3-Sasakian case. Can we prove that $\mathcal{M}_{SDCI}$ is hyperKhler?

7. **(Jesse Madnick)** Can we say something about the non-zero torsion classes of a G_2-structure, the appearance of which will be a necessary and sufficient condition for every G_2-instanton being a Yang–Mills connection?

8. **(Gonalo Oliveira)** There is a result of Derdzinsky from the 80s which says that "(M^4, g, ω) extremal (i.e., ∇S is a holomorphic vector field with S the scalar curvature) and g Bach-flat $\implies (M^4, S^{-2}g)$ is Einstein."

   Can we find conditions on (N^5, g, η, Φ) which is a Sasakian 5-manifold and is extremal, analogous to Bach-flatness in the 4-dimensional case, which would imply the existence of a conformal metric which is Einstein?

9. **(Spiro Karigiannis)** Let α be a calibration k-form on $\mathbb{R}^n$ equipped with the standard metric and orientation. (That is, α has constant coefficients and comass one.) Let $G = \mathrm{Stab}_{\mathrm{O}(n)}\alpha$. There are several properties that α may or may not have. These are the following:

   (a) G acts transitively on the unit sphere S^{n-1} in $\mathbb{R}^n$.

   (b) G acts transitively on the Stiefel manifold $V_{r,n}$ of r-tuples of orthonormal vectors in $\mathbb{R}^n$ for $1 \leq r \leq k-1$. (Note that (a) is just (b) for $r = 1$.)

   (c) G acts transitively on the Grassmanian Gr_α of α-calibrated k-planes in $\mathbb{R}^n$

   (d) Let W be an α-calibrated k-plane in $\mathbb{R}^n$, and let $H = \{P \in G : P(W) = W\}$ be the stabilizer in G of W. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G, H respectively. Then we can write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp_\mathfrak{g}}$. We always have the equality $\mathfrak{h} = \Lambda^2(W) \oplus \Lambda^2(W^\perp)$ and the inclusion $\mathfrak{h} \supseteq \mathfrak{g} \cap (W \otimes W^\perp)$. Property (d) is that the inclusion is an equality. We say such an α is *compliant*. If property (c) holds, then property (d) is independent of the choice of $W \in \mathrm{Gr}_\alpha$.

   (e) Suppose that (c) holds. Let $W \in \mathrm{Gr}_\alpha$. Let $e_1, \ldots, e_k$ be an oriented orthonormal basis of W and let $\nu_1, \ldots, \nu_{n-k}$ be an oriented orthonormal basis of $W^\perp$. Then in terms of the decomoposition $\Lambda^k(\mathbb{R}^n) = \Lambda^k(W \oplus W^\perp) = \oplus^k_{p+q}\Lambda^p(W) \otimes \Lambda^q(W^\perp)$, we can write $\alpha = \sum_{p+q=k} \alpha_{p,q}$. Property (e) is that only even values of q occur in this decomposition. One can show that (e) implies (d).

   (f) For any $v \in S^{n-1}$, both $v \lrcorner \alpha$ and $v \lrcorner \star \alpha$ have comass one. This is equivalent to the fact that any unit vector v lies in an α-calibrated k-plane and also lies in a $(\star\alpha)$-calibrated $(n-k)$-plane. We say that such an α is *rich*.

   (g) For $v \in S^{n-1}$, let $L_v = \mathrm{Span}\{v\}$, so $\mathbb{R}^n = L_v \oplus L_v^\perp$. Write $\alpha = v \wedge \beta_v + \gamma_v$ where $v \lrcorner \beta_v = v \lrcorner \gamma_v = 0$, so $\beta_v \in \Lambda^{k-1}(L_v^\perp)$ and $\gamma_v \in \Lambda^k(L_v^\perp)$. Property (g) is that $\langle \beta_v, w \lrcorner \gamma_v \rangle = 0$ for all $w \in \mathbb{R}^n$ and all $v \in S^{n-1}$.

   For some mysterious reason, every single one of the above properties is satisfied by the interesting geometric calibration forms (Kähler, special Lagrangian, associative, coassociative, Cayley). Several of these properties are in some sense quantifying that there are many α-calibrated k-planes. Is there a single property that a calibration α could have which implies all of these? If so, what is the geometric significance of such a property?

10. **(Daniel Platt)** Let $s : \mathbb{T}^3 \to X^{4k}$ (hyperKhler) and let $\{x_1, x_2, x_3\}$ be coordinates on $\mathbb{T}^3$ and I_1, I_2, I_3 be the triple of complex structures on X^{4k}. The Fueter operator on s is $Fs = \sum_{i=1}^3 I_i \left(ds(\frac{\partial}{\partial x_i}) \right)$. If $Fs = 0$ then s is called a Fueter section. A known fact about F is that it is an index 0 operator and hence the expectation is that Fueter sections are rigid. However, all know examples of X^{4k} where s is explicit have moduli. So can we have Fueter sections which do not have moduli, i.e., that are rigid?

11. **(Jason Lotay)** Suppose $E \to (M^7, \varphi)$ with M being a G_2-manifold. Suppose A is a G_2-instanton on E. What does it tell us about E? The situation we have in mind is that of bundles over Khler manifolds, where existence of Hermitian–Yang–Mills connection $\iff$ the bundle is stable due to Donaldson–Uhlenbeck–Yau theorem or the Kobayashi–Hitchin correspondence. So the questions are 1) Is there a notion of stability of bundles E over a G_2-manifold? 2) Is there a Donaldson–Uhlenbeck–Yau/Kobayashi–Hitchin correspondence type theorem?

The loop space of a G_2 manifold is a Calabi–Yau manifold. Can we use this information and point of view to get a notion of stability and answer above questions?

Is there a Geometric Invariant theory, moment map and/or symplectic reduction picture associated with the not-yet-defined notion of stability?

12. **(Spiro Karigiannis)** In the Khler case, $\Omega^2 = \Omega^{2,0} \oplus \Omega^{0,2} \oplus C^\infty\omega \oplus \Omega_0^{1,1}$ and a connection A is Hermitian–Yang–Mills $\iff F_A \in \Omega_0^{1,1}$. Now consider a gerbe over (M^7, φ) and let A be a connection on the gerbe. Then F_A is a 3-form on M. Is it true that F_A is "Hermitian–Yang–Mills" $\iff F_A \in \Omega^3_{27}$?

13. **(Gavin Ball)** Consider the standard G_2-structure φ on $\mathbb{R}^7$ and let S be the set of degenerate 3-forms on $\mathbb{R}^7$. S is singular. What can we say about dist(φ, S), the distance between φ and S? We know that dist$(\varphi, S) \leq 1$ but can it be smaller?

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