Set Theoretic Topology (23w5025)

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1 Overview of the Field

Topology has for decades served both as a source of inspiration and a testing ground for novel set-theoretic techniques and axioms. The workshop proposes to explore the mutual interrelations between topology, set theory and algebra while paying special attention to *homogeneous*[1] spaces. Homogeneity is a critical property of topological groups and connected manifolds.

There were several major developments concerning homogeneous spaces in the near past. Hoehn and Oversteegen [15] famously solved an ancient problem traceable to Knaster and Kuratowski by classifying homogeneous compacta in the plane:

- ([15]) *Every homogeneous compact subset of the plane is either finite, homeomorphic to the Cantor set or homeomorphic to the product* $Z \times K$ *where* Z *is either finite or homeomorphic to the Cantor set and* K *is the circle, pseudo-arc, or the circle of pseudo-arcs.*

and Dow and Tall [11] proved

- ([11]) *It is consistent that every hereditarily normal manifold of dimension larger than* 1 *is metrizable.*

In the past decade sophisticated set-theoretic methods have been successfully used to solve long-standing problems in topological algebra

- ([27]) *Every countable Fréchet group with an analytic topolgy is metrizable.*
- ([17]) *It is relatively consistent with* ZFC *that every separable Fréchet topological group is metrizable.*
- ([25]) *It is relatively consistent with* ZFC *that every sequential topological group is metrizable or has sequential order* ω_1.
- ([16]) *Countable compactness is not productive in the class of topological groups.*

An exciting recent development creates a strong link between set theory, algebraic topology, and homological algebra. It has long been desired to create a robust homology theory for locally compact separable metric spaces. A natural candidate for such a theory was the *strong homology*. In the 1980's Mardešić and

[1] A topological space is *homogeneous* if given two of its points there is a homeomorphism of the space sending one to the other.

Prasolov [21] noticed that strong homology, though strong shape invariant, consistently fails to be additive and compactly supported (see also [10]) even when restricted to locally compact subsets of $\mathbb{R}^3$. Very recently Bannister, Bergfalk and Moore [1] (based on also recent work of Bergfalk and Lambie-Hanson [2]) announced that

- ([1]) *Assuming the consistency of a weakly compact cardinal, it is consistent that strong homology is compactly supported, and therefore additive, on the class of locally compact separable metric spaces.*

Finally, the workshop will include the study of topological games. Topological games have been used to characterize generalized completeness properties and selection principles on topological spaces. In addition to topological applications, there is active research on partial information strategies of players in the games. Recently, Brian, Dow, Milovich, and Yengulalp [4] gave a consistent solution to a long-standing question of Telgársky [26] by proving that

- ([4]) *It is consistent that if NONEMPTY has a winning strategy in the Banach-Mazur game on a regular space X, then she has a winning 2-tactic.*

A statement of the objectives of the workshop and an indication of its relevance, importance, and timeliness

The results mentioned in the previous section are examples of deep and fruitful interactions between set-theory and topology, and the relatively less explored connection with algebra, which open room for new interactions. Sophisticated set-theoretic techniques of descriptive set-theory, forcing, large cardinals, and coding using combinatorial guessing and coherence principles were present in the proofs of the above results. The workshop aspires to bring together experts and young talented researchers, especially Mexican and Latin American, in the areas of topology, set theory, and topological algebra to take advantage of different perspectives and expertises to further develop these connections and explore new ones.

A primary objective of the workshop is to advance the study of homogeneity phenomena in topology. In [5] one finds a recent survey on cardinal functions on homogeneous spaces, [22] deals with rigidity and homogeneity properties of metric spaces, as does breakthrough work of Dijkstra and van Mill ([7]) which gives a topological characterization of the Erdős space (see also [12]). A very active area of research deals with *countable dense homogeneity* of topological and metric spaces (see e.g. [14, 13]) traceable back to the orgins of dimension theory in the works of Brouwer and Fréchet. A large and important class of homogeneous spaces comes from topological algebra and functional analysis; [8] studies countable dense homogeneity of topological vector spaces. Countable dense homogeneity has also been shown to be closely related to combinatorial properties of filters, see [20]. Questions arising in functional analysis also motivated a recent metrisation theorem ([9]) of compact spaces with $\mathbb{P}$-diagonal, while [19] gives a characterization of the universal separable Banach space, the so called, Gurarij space in terms of topological games. Topological games are also studied in [28], bridging a gap in some generalized completeness properties: "domain representability" which has its roots in theoretical computer science and the property of subcompactness of de Groot [6].

The workshop proposes to identify the fundamental problems of topological algebra whose solution likely requires deep set-theoretic methods in discussion between experts in both fields, and start a collaboration towards solving them. The recent advances in the study of convergence properties of topological groups ([17, 25, 27] is being synthesized into a paradigm set-theoretic axiom for the study of convergence in algebraic-topological structures in [18].

The vastly untouched area between set-theory, homological algebra and algebraic topology is to be explored during the workshop. Understanding this area seems to require development of new set-theoretic methods related to coherence, in general, and Todorčević's method of *minimal walks on ordinals*, in particular. The subject seems to require a thorough study of higher dimensional analogues of these ideas, as well as better understanding of combinatorics of small uncountable cardinals beyond $\aleph_1$ (see [1, 2]). Another promising line of research in the same area (see [3]) proposes the use of descriptive set-theoretic methods in homology theory by considering a 'continuous (or definable) (co-)homology", refining the methods of standard (co-)homology theory by considering the (co-)homology groups as topological groups and the morphisms appearing in the corresponding exact sequences respecting their topological/Borel structure.

This workshop comes at an opportune time, taking advantage both of the above outlined recent major advances in these areas, the new connections they establish, and the interest these advances naturally generate, as well as the geographical location of CMO, offering a unique opportunity to bring together young dynamic group of Mexican and Latin American mathematicians with leading experts in the areas involved.

The program of the meeting would offer lectures outlining major recent advances and directions for further research, two mini-courses, one of them dedicated to the connection between set-theory and algebraic topology and problem sessions devoted to identifying further directions of research of common interests to the mathematical areas involved.

Another one of the objectives of the workshop will be to generate research questions and projects for students, including undergraduates. We plan to dedicate one afternoon to a mentoring event moderated by Lynne Yengulalp for faculty at teaching institutions dedicated to the development of topics and styles of questions that are suitable for undergraduate research projects. The rationale is to foster recruitment of students, especially students from under-represented groups, to pursue graduate studies in set-theoretic topology as well as to broaden our community by supporting researchers at teaching institutions.

We are prepared, in case of unforeseen circumstances, to hold the meeting in a virtual form during the same week it would be assigned. Such a virtual meeting would be held under the auspices on the UNAM, Mexico, and would use its infrastructure and electronic services to connect the participants.

Press Release

The workshop is designed to explore the interactions that exists between set theory, topology, and algebra. The focus of the workshop will be on the study of topological problems of set-theoretic flavor arising in topological algebra and algebraic topology, as well as the study of topological games and homogeneity.

The program will bring together senior leaders in the field and students and junior researchers, specifically young mathematicians from Latin America. We proudly acknowledge the growing number of strong female mathematicians in the area, a trend the list of participants of the workshop will certainly reflect.

2 Activities and progress made during the workshop

2.1 General structure

The workshop was organized with the intent of providing participants with up to four talks a day, hour long each, in hybrid format while leaving ample time for in depth discussion in working groups. In total there were 15 talks, 12 of them presented on site and 3 of them presented remotly. An hour on the Monday, Tuesday and Thursday evenings was devoted to problem sessions. Participants volunteered to present problems they found interesting at the blackboard and say a few words of background. In many cases audience members were able to provide further enlightening comments. Two hours at the end of the workshop on Friday were dedicated to discussions concerning undergraduate teaching and a possibility for creating a database of problems suitable for student projects.

2.2 Schedule of the Workshop: Set Theoretic Topology

Monday, July 31

09:30 - 10:30 Will Brian: The Borel partition spectrum.

11:00 - 12:00 Hayden Pecoraro: A countable mH-separable Fréchet space that is not H-separable.

12:30 - 13:30 Work in groups

15:00 - 16:00 Santi Spadaro: Infinite games and cardinal functions.

16:30 - 17:30 Roy Shalev: A minimal non-σ-scattered linear order of an inaccessible cardinality.

18:00 - 19:00 Problem session

Tuesday, August 1

09:30 - 10:30 Jeffrey Bergfalk: A descroptive approach to manifold classification.

11:00 - 12:00 Chris Lambie-Hanson: Condensed mathematics, extremally disconnected spaces, and forcing.

12:30 - 13:30 Work in groups

15:00 - 16:00 Jorge Antonio Cruz Chapital: Transferring structures from ω to ω_1.

16:30 - 17:30 Natasha Dobrinen: Cofinal types of ultrafilters on measurable cardinals.

18:00 - 19:00 Problem session

Wednesday, August 2

09:30 - 10:30 Hector Barriga Acosta: MH and Δ.

11:00 - 12:00 Todd Eisworth: Weakly precipitous ideals and the Raghavan-Todorcevic partition theorem.

12:30 - 13:30 Work in groups

Thursday, August 3

09:30 - 10:30 Mirna Dzamonja: Changing logic to capture convergence.

11:00 - 12:00 Joerg Brendle: Higher dimensional cardinal characteristics.

12:30 - 13:30 Work in groups

15:00 - 16:00 Iván Sánchez: Hattori topologies on almost topological groups.

16:30 - 17:30 Lyubomyr Zdomskyy: Ideals and weakenings of the Fréchet-Urysohn property in function spaces.

18:00 - 19:00 Problem session

Friday, August 4

09:30 - 10:30 Andrea Medini: Zero-dimensional σ-homogeneous spaces.

11:00 - 13:00 Student research discussion

2.3 Summary of the talks

Both the opening lecture by Will Brian and the lecture of Todd Eisworth dealt with the structure of the most important topological space - the *real line* $\mathbb{R}$. A classical theorem of Hausdorff states that the real line can always be partitioned into ω_1-many disjoint Borel sets. The question considered by Will Brian was what is/consistently can be the *Borel partition spectrum* of the real line, i.e. the set of uncountable cardinals κ such that the real line can be partitioned into κ-many disjoint dense sets. Such a spectrum has to contain ω_1 by the result of Hausdorff, obviously, the cardinal $\mathfrak{c} = |\mathbb{R}|$, and is closed under singular limits. Surprisingly, assuming that $0^\dagger$ does not exist, also successors of singular cardinals of countable cofinality which are in the spectrum belong to the spectrum. Finally, a forcing construction for countable spectra satisfying these restrictions was given.

Todd Eisworth thoroughly analyzed the recent celebrated solution of Raghavan and Todorcevic to the *Galvin conjecture*: if we color the pairs from an uncountable set of reals, then we can find a set homeomorphic to the rationals on which the coloring assumes at most two values. Raghavan and Todorcevic's proof requires the existence of a Woodin cardinal. Eisworth's analysis of the game associated to precipitous ideals reduced the large cardinal hypothesis to the existence af a Ramsey cardinal. It remains an open problem if any large cardinal assumption is needed for the solution of the problem at all.

In another impressive work on linearly ordered topological spaces Roy Shalev, building on a very recent work of Cummings, Eisworth and Moore who gave consistent examples of a minimal non-σ-scattered linear orders of size an arbitrary successor cardinal, presented his construction of such examples for inaccessible cardinals.

Hector Barriga Acosta's talk reported on the progress made by him, Will Brian and Alan Dow, on one of the oldest open problems in topology - Tietze's *normal box-product problem* which in its simplest form asks whether the box-product of countably many copies of $\mathbb{R}$ is normal. To deal with the problem Judy Roitman introduced two combinatorial principles - MH (the *model hypothesis*) and its weakening Δ, simply called *delta*, sufficient for a positive answer to the problem. This very interesting talk offered solutions to several problems posed by Roitman concerning these by showing (1) that the model hypothesis can consistently fail in a model where Δ holds and, in particular, the two principles are not equivalent and (2) presenting a general preservation scheme which hints that perhaps "killing" Δ might be very difficult. These results offer a major new insight into this classical problem.

The lectures by Hayden Pecoraro and Lyubomyr Zdomskyy talked about the important class of *Fréchet-Urysohn spaces*, i.e. spaces where closure operator is simply described by collecting the limit points of convergent sequences, in the context of *selection principles* introduced by Scheepers. Two of such selection properties are H- and mH-separability. While all countable Fréchet-Urysohn spaces are necessarily mH-separable, recent work of Bardyla, Maesano, and Zdomskyy showed that consistently there is a countable Fréchet-Urysohn space that is not H-separable. Pecoraro sketched a construction of such a space from substantially weaker assumptions $\mathfrak{p} = \mathfrak{b}$ or $\mathfrak{b} = \mathfrak{c}$.

Zdomskyy's talk reported on joint work with Bardyla and Šupina dealing with the Fréchet-Urysohn property in the realm of spaces of continuous functions, studying, in particular, properties between the Fréchet-Urysohn and Pytkeev. Using the so-called local-global duality in C_P-theory these were re-cast into properties of the base spaces and analyzed by the means of ideals on natural numbers as parameters using the Katětov preorder on them.

Convergence was the leading theme of an inspiring high level talk by Mirna Džamonja who shared her insight into topological content of subjects of modern logic ranging from ultraproducts to various types of infinitary logic to Homotopy Type Theory tying them unexpectedly with deep combinatorial statements such as *Szemeredi's regularity lemma* and measurable combinatorics of graphons.

Hattori's study of topologies intermediate between the Euclidean and Sorgenfrey topologies on the real line was the source of inspiration for the work of Ivan Sanchez on almost topological groups (i.e. groups which are topological except for the inverse not being continuous) and their reflecting topological groups. The talk presented a large number of interesting results concerning e.g. second-countability, network weight, metrizability and local compactness, and open problems on the subject.

A study of Todorcevic's *construction schemes* was presented by Jorge Cruz Chapital reporting on joint work with Osvaldo Guzmán and Stevo Todorcevic. The construction schemes are *morass*-like structures designed to buld uncountable object from a coherent family of its finite approximations. The talk using a very graphical presentation clearly conveyed the range and elegance of the applications of this novel method.

The talk of Santi Spadaro surveyed the results of several of his joint papers dealing with the impact of infinite games on the theory of cardinal functions in topology.

Both consistency and ZFC results concerning higher dimensional analogues of the cardinal characteristics from the *Cichoń ́s diagram* were presented by Joerg Brendle. His talk focused on whether the duality phenomena present in the Cichoń diagram persist in the higher dimensional context, giving both positive and negative results.

The talks by Jeffrey Bergfalk and Chris Lambie-Hanson offered novel connections of set theory and set-theoretic topology to other areas of mathematics. Jeffery Bergfalk talked about the attempt at incorporating the *classification problem for topological/differentiable manifolds* into the general framework of *Borel equivalence relations* - a joint effort with Ian Smythe - and outlined an approach to coding the isomorphism relation between various types of manifolds by a a definable relation on a Polish space, and giving both lower and upper bounds on their complexity in the Borel-reducibility order.

Chris Lambie-Hanson prepared an excellent exposition into the set-theoretic and topological aspects of Dustin Clausen and Peter Scholze's *Condensed mathematics* making this exciting, important and very abstract new development in mathematics more accessible to set theorists and topologists. He has isolated problems and connections with the study of forcing and of compact extremally disconnected topological spaces as well as coherence properties of ultrafilters of interest to and motivated by the works of Clausen and Scholze. This is a new connect bringing together researchers from areas that has historically drifted far apart and has attracted interest and need for collaboration from both sides.

The realization that the well studied *Galvin's property* for ultrafilters over uncountable sets is equivalent to not being Tukey maximal started the joint reserach of Natasha Dobrinen and Tom Benhamou on the Tukey types of ultrafilters over measurable cardinals, presented by Natasha Dobrinen. She pointed out that the situation for ultrafilters on measurable cardinals turns out to be quite different from that on a countable set, sometimes greatly simplifying the situation and sometimes posing new obstacles.

The concluding lecture by Andrea Medini presented a descriptive set theoretic extension (joint with Zoltán Vidnyánszky) to a result of Ostrovsky that every zero-dimensional Borel space is σ-homogeneous to classes beyond the Borel hierarchy. They showed that (1) Assuming the Axiom of Determinacy, every zero-dimensional space is σ-homogeneous, (2) Assuming the Axiom of Choice, there exists a zero-dimensional space that is not σ-homogeneous, and (3) in the constructible universe L, there exists a co-analytic zero-dimensional space that is not σ-homogeneous. The talk was a fitting culmination of the scientific program of the workshop linking in an essential way the two of the main threads of the workshop, games and homogeneity

2.4 Open Problems presented at the problem sessions

Evenings of Monday, Tuesday and Thursday were devoted to long problem sessions where both new and old questions were beind discussed. Here we shall mention some of the problems posed and discussed.

Paul Szeptycki informed us about his fascinating ongoing joint project with Arkady Leiderman seeing old properties of sets of reals involved in the *normal Moore space* program from the 80's resurface in a new and surprising connection with well studied properties of function spaces such as Grothendieck's property of being *distinguished*. Recall that a *Q-set* is an uncountable subset of $\mathbb{R}$ such that every subset is a G_δ, while a *Q-space* is a topological space which is not σ-discrete such that every subset is a G_δ. A Δ-*set* is an uncountable subset of $\mathbb{R}$ such that for every sequence $(U_n)_{n\in\omega}$ such that $\bigcup_{n\in\omega} X_n = X$ there is $X_n \subseteq U_n$ open such that $\{U_n : n \in \omega\}$ is point-finite.

The class of all Δ-spaces consists precisely of those spaces X for which the locally convex space $C_p(X)$ is distinguished.

The questions of interest in this investigation (some of them old some new) are:

- (Balogh) Is there a Lindelöf Q-space in ZFC?
- Is there, consistently, a Δ-set whose square is not a Δ-set?

- Does the existence of a Δ-set imply that $2^{\aleph_0} = 2^{\aleph_1}$?
- Is every compact Δ-space a countable union of Eberlein compact spaces?

Cristina Villanueva offered a possible set-theoretic take on the more than a 130 year old *Toeplitz' conjecture* asking whether every simple closed curve in the plane contain all four vertices of some square. For that she called a set $A \subseteq \mathbb{R}^2$ is a *square catcher* if A contains at least one vertex of each square of $\mathbb{R}^2$ and asked about topological and measure theoretic behavoiur of square catchers.

Jorge Cruz Chapital defined a topology on function spaces intermediate between the pointwise and the uniform one: Given a space X, $f \in C(X)$, $F \subseteq X$ closed and separable and $\epsilon > 0$ let

$$V(f, F, \epsilon) = \{g \in C(X) \mid \forall x \in F(|g(x) - f(x)| < \epsilon)\}.$$

Consider $C(X)$ equipped with the topology generated by sets of the form $V(f, F, \epsilon)$ and denote it this space by $C_s(X)$.

He mentioned his recent results with Tamariz, Villegas and Rojas:

- $c(C_s(X)) = \omega$ if and only if X is compact and metrizable.
- $(\neg\mathsf{CH})$ Let X be a set of ordinals. If $c(C_s(X)) = \omega_1$ then X is pseudo-compact.
- If $X \subseteq \alpha^n$ and X is countably compact then $c(C_s(X)) = \omega_1$.

and asked about the cellularity of $C_s(\beta\mathbb{N})$.

Michael Hrušák (in a joint work with Fernando Hernández and Norberto Rivas) proposed a topological study of (definable) ideals on countable set. Given and ideal $I \subseteq \mathcal{P}(\omega)$ consider the *bounded topology* τ_{bd} as the strongest topology on I for which every bounded sequence convergent in the product topology τ converges. He noted that

- $\tau = \tau_{bd}$ if and only of I is a non meager P-ideal.
- (I, τ_{bd}) is Fréchet-Urysohn if and only if I is P-ideal.
- (I, τ_{bd}) is Polish if and only if I is an analytic P-ideal.
- $(\mathsf{FIN} \times \mathsf{FIN}, \tau_{bd})$ is not regular.
- $(I_{\frac{1}{n}}, \tau_{bd}) \approx \mathfrak{E}_c$ (the complete Erdős space).

He then asked:

1. Which ideals are regular in τ_{bd}?
2. Is (I, τ_{bd}) metrizable iff I is a P-ideal?
3. Is (I, τ_{bd}) Lindeloff iff I has no uncountable strongly unbounded subsets?
4. Assume I is an analytic P-ideal. Is (I, τ_{bd}) homeomorphic to one of $\omega, 2^\omega \setminus \{\emptyset\}, \omega^\omega, \mathfrak{E}_c, \mathfrak{E}_c^\omega$?

Lyubomyr Zdomskyy recalled the related famous open problem from set-theory of the reals

- Is there a non-meager P-filter in ZFC?

and asked its variation:

- (Zdomskyy) Is there a filter $\mathcal{F}$ such that $\mathcal{F}^+$ is Menger?

Recall that X is *Menger* if for every sequence $\{\mathcal{U}_n \mid n \in \omega\}$ of open covers of X there is a sequence $\{\mathcal{V}_n \mid n \in \omega\}$ such that each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n\in\omega} \bigcup \mathcal{V}_n$. Note that If $\mathcal{F}^+$ is Menger then $\mathcal{F}$ is non-meager and P. A weaker question is:

- (Zdomskyy) Is there a ZFC example of a semifilter such that $\mathcal{F}^+$ is Menger and $\mathcal{F} \subseteq \mathcal{F}^+$?

Here $\mathcal{F}$ is a *semifilter* if $\mathcal{F}$ is closed under finite modifications of its elements and under taking supersets.

Large part of the second session was devoted to Ramsey theoretic questions. First Keegan Dasilva asked questions about categories $\mathcal{K}$ of finite relational structures. Call $\mathcal{K}$ *Ramsey* if for all $A, B \in \mathcal{K}$ and $k \in \omega$ there exists $C \in \mathcal{K}$ such that for all $c : HOM(A, C) \to k$ there exists $f \in HOM(B, C)$ such that $|c[f \circ HOM(A, B)]| \leq 1$. A *preadjuntion* from a category $\mathcal{K}_1$ to $\mathcal{K}_2$ is a pair (F, G) such that $F : \mathcal{K}_1 \to \mathcal{K}_2$ and $G : \mathcal{K}_2 \to \mathcal{K}_1$ and $\forall (A, B) \in \mathcal{K}_1 \times \mathcal{K}_2 \exists \Phi_{(A,B)} : HOM(F(A), B) \to HOM(A, G(B))$ such the corresponding diagrams commute. Then we say that $\mathcal{K}_2 \leq_T \mathcal{K}_1$.

- (Masulovic) $\mathcal{K}_2 \leq_T \mathcal{K}_1$ and $\mathcal{K}_2$ is Ramsey, then $\mathcal{K}_1$ is Ramsey.

He noted that Dual Ramsey $\equiv_T$ Graham-Rothschild and asked:

- (Masulovic) Dual Ramsey $\leq_T$ Ramsey?
- If $\mathcal{K}$ is "nice enough" is it $\leq_T$ Ramsey?

Then David Fernández Bretón talked about Orwing's problem and about the sum-sets structure of the reals. To fix a notation, consider a commutative semi-group S. We write $S \to (\kappa)^+_\theta$ to denote that for every $c : S \to \theta$ there is $X \in [S]^\kappa$ such that $X + X$ is monochromatic. Orwing's original question was:

- Does $(\mathbb{N}, +) \to (\omega)^+_2$?

Partial solutions and related results were given by Hindman, and Hindman-Leader-Strauss:

- (Hindman) It is false that $(\mathbb{N}, +) \to (\omega)^+_3$.
- (Hindman-Leader-Strauss) There is θ such that if $\mathfrak{c} = \omega_2$ then it is false that $(\mathbb{R}, +) \to (\omega)^+_\theta$.
- (Hindman-Leader-Strauss) $(\mathbb{R}, +) \to (\omega)^+_2$.

The natural question is then:

- What is the optimal θ in the Hindman-Leader-Strauss' theorem?

Michael Hrušák and Paul Szeptycki mentioned questions concerning products of Fréchet-Urysohn groups:

- (Hrušák-Shibakov) Is there a Fréchet-Urysohn topological group G such that $G \times G$ is not Fréchet-Urysohn?
- (Peng-Todorcevic) Suppose that either MA + ¬CH or PFA holds. Are there subsets X_1 and X_2 of the real line such that both $C_p(X_1)$ and $C_p(X_2)$ are Frćhet-Urysohn but $C_p(X_1 \times X_2)$ is not?

Rodrigo Hernández recalled questions on functionally countable spaces. Recall that a topological space X is *functionally countable* if $|f[X]| \leq \omega$, for all continuous functions $f : X \to \mathbb{R}$.

Examples of functionally countable spaces include all countable spaces and ordinals, Lindelöf P-spaces and Lindelöf scattered spaces. Also a compact space X is functionally compact if and only if X is scattered (Rudin). He mentioned that

- (L. E. Gutiérrez-Dominguez, R. Hernández-Gutiérrez) Let X be a Suslin line. Then there exists $Y \subseteq X$, dense in X, such that Y is functionally countable but $Y^2 \setminus \Delta$ is not.

and asked:

- Is the existence of a Suslin line X such that X^2 is functionally countable consistent?
- Are there consistently Suslin lines X and Y such that $X \times Y$ is functionally countable?

- Is there a functionally countable Aronszajn line?

Carlos López Callejas told us about problems raised by Kubiś and Szeptycki that he has been working on together with César Corral and Osvaldo Guzmán.

Let X be a topological space and $n \geq 1$. Given $p \in X$ and $B \in [\omega]^\omega$ a function $f : [B]^n \to X$ is said to *converge* to p if for every open neighborhood V of p there is an $m \in \omega$ such that $f[[B \setminus m]^n] \subseteq V$. The space X is *n-sequentially compact* if for all $f : [\omega]^n \to X$ there is a $B \in [\omega]^\omega$ such that $f \restriction [B]^n$ converges in X.

- Given $n \geq 1$, is there a n-sequentially compact space X which is not $(n+1)$-sequentially compact space?
- Is there a Ψ-space with these properties?

The answer to the question is positive if CH holds (Kubiś), or if $\mathfrak{b} = \mathfrak{c}$ or $\mathfrak{b} = \mathfrak{s}$ (Corral, Guzmán, López-Callejas). It is also open if the question has a positive answer if $\mathfrak{s} \leq \mathfrak{b}$.

Finally, Will Brian recalled the open *Toronto space problem*: A *Toronto space* is an uncountable non-discrete Hausdorff topological space homeomorphic to all of its uncountable subspaces. The problem is

- Is the existence of a Toronto space consistent ?

While it is unknown whether such a space can exist, one can quickly deduce that such a space would have to be scattered of height ω_1 with countable levels. What is not even clear at the moment is if such a space has to be regular. What can be deduced is that one of the following happens:

1. X is a regular space.
2. If I_0 is the subset of isolated points of X, i.e. its first level, then $\{p \in X \setminus I_0 \mid X\ is\ T_3\ at\ p\}$ is a closed subset of I_1 (the second level of X), and for all $p \in X$ of level α there exists and open set U such that $p \in U$ and $\overline{U} \subseteq \bigcup_{\beta<\alpha} I_\beta$.
3. Every open neighborhood of every non-isolated point has countable closure.

- (Brian) Is one of the three possibilities false?
- Does Martin's Axiom imply that there are no Toronto spaces?

2.5 The Undergraduate Teaching mentoring event

The participants of BIRS-CMO event Set-Theoretic Topology 23w5025 had a lively discussion about set-theory and topology projects for students at the pre-PhD level. Nathan Carlson (California Lutheran University), Steven Clontz (University of South Alabama), and Santi Spadaro (University of Palermo) presented student projects that they had led and then Lynne Yengulalp (Wake Forest University) moderated a problem session style discussion of other student projects.

Presenters explained student research topics and provided strategies for supporting students at different levels and backgrounds. The result of the discussion is a diverse list of projects and project ideas (found here: https://sites.google.com/wfu.edu/topology-projects-for-students/) that will be maintained for the use of the set-theoretic topology community. In addition to student research ideas, participants were also introduced to https://topology.pi-base.org/ which is an open source database of topological spaces, theorems, and counterexamples that can be the starting point for student contributions to the area.

3 Outcome of the Meeting

The meeting brought together experts from the area of set-theoretic topology and graduate students and post-docs who represented more than half of the 42 participants present at Casa Matemática Oaxaca. Very good talks were given by students and early career researchers (Barriga, Cruz Chapital, Pecoraro, Shalev).

There were 3 talks presented remotely by ZOOM by Natasha Dobrinen (University of Notre Dame), Andrea Medini (University of Vienna) and Iván Sánchez (Universidad Autónoma Metropolitana) and other researches participated in the activities via ZOOM.

Aside from the 15 lectures presented, the meeting offered ample space and time devoted to discussions in person or in small groups. This allowed for the students to get important in-person input on their research projects from leading experts in the field, discuss existing research projects and start new ones. In particular, there have been groups discussing the connections of topological games to box products and functional analysis (proximal games of Jocelyn Bell), discussions on the Ramsey Theory and higher dimensional convergence (Szeptycki, Guzmán), and coherent sequences indexed by ultrafilters (Bergfalk, Dow, Lambie-Hanson).

There was substantial progress reported on old and important problems in the area (Barriga, Eisworth) and new and promising directions of research in set-theory and topology were outlined both in the talks and in the very active problem sessions.

Exciting new interactions of these areas with other areas of mathematics have been proposed, in particular, in the work of Jefferey Bergfalk, Chris Lambie-Hanson and others which has profound connection with geometry, homological algebra, algebraic topology and the fast developing condensed mathematics.

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