# The André-Oort conjecture - an overview.

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Motivational example :

## Theorem ("Manin-Mumford conjecture")

*Let $A = \Lambda \backslash \mathbb{C}^g$ be an abelian variety of dimension $g$. Let $Z \subset A$ be a subvariety containing a Zariski dense set $\Sigma$ of torsion points. Then $Z$ is a translate of an abelian subvariety by a torsion point.*

Torsion points are also called **special** and translates of abelian subvarieties by torsion points **special subvarieties**.
Equivalently a subvariety of $A$ contains a *finite* number of *maximal* special subvarieties.
The **weakly special subvarieties** - translates of abelian subvarieties by arbitrary points - also play an important role.
The statement is motivated by the Mordell-Lang conjecture (which contains the Mordell's conjecture).

This theorem has a large number of **very** different proofs since 1983 (the first proof was given by Michel Raynaud).
The one that is most relevant to us is the one by Pila-Zannier (2006) that uses o-minimality and functional transcendence.

## Conjecture (André-Oort, theorem ( ?) 2021)

*Let $S$ be a Shimura variety and $\Sigma \subset S$ a set of special points. Irreducible components of the Zariski closure of $\Sigma$ are special subvarieties.*

**Latest result :** the conjecture holds in full generality (via o-minimal approach).

**The first nontrivial case :** the case of a product two modular curves. The statement is that if a curve $C$ in $\mathbb{C} \times \mathbb{C}$ contains an infinite set of special points (pairs of CM elliptic curves) and both projections are dominant, then $C = Y_0(n)$ for some $n$ (equivalently, defined by a modular polynomial $\Phi_n$).

Bas Edixhoven proved this under GRH in 1996, his method generalised to the general case (under GRH). J. Pila proved it in 2006 unconditionally using the o-minimal theory and functional transcendence that was a starting point.

The Pila-Zannier approach can be summarised thus.
Let

$$\pi : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{C} \times \mathbb{C}$$

and $\mathcal{F} \times \mathcal{F} \subset \mathbb{H} \times \mathbb{H}$ the usual fundamental domain.
Let $C$ be a curve in $\mathbb{C} \times \mathbb{C}$ containing an infinite set of special points.
One shows that

- $\pi$ is definable in an o-minimal structure ($\mathbb{R}_{an,exp}$ in this case) when restricted to $\mathcal{F} \times \mathcal{F}$.
- The height of 'pre-special points' is bounded in terms of the 'discriminant of the special points'.
- The Galois orbits grow as a power of the discriminant (easy in this case).
- Pila-Wilkie counting theorem implies the existence of a positive dimensional semi-algebraic in the preimage of $C$
- One concludes using a functional transcendence result.

# o-minimality

A **structure** $S$ over $\mathbb{R}$ is a collections of subsets $S_n$ of $\mathbb{R}^n$ for each $n \geq 1$ such that

1. $S_n$ contains all semialgebraic sets, in particular $\emptyset$ and $\mathbb{R}^n$ are in $\mathbb{R}^n$ for each $n$.
2. If $A, B$ are in $S_n$, then $A \cup B$ and $A \cap B$ are in $S_n$ and $\mathbb{R}^n \backslash A$.
3. If $A \in S_n$ and $B \in S_m$, then $A \times B$ is in $S_{n+m}$.
4. Let $A \in S_{n+m}$ and $p \colon \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ be the projection. Then $p(A) \in S_n$.

A subset $A$ of $\mathbb{R}^n$ is definable in $S$ is $A \in S_n$.
A structure is called o-minimal if the only definable sets in $\mathbb{R}^1$ are finite unions of points and open intervals i.e. exactly the semialgebraic sets.
Let $A \in S_n$ and $B \in S_m$. A function $A \longrightarrow B$ is definable in $S$ if its graph in $A \times B$ is definable (i.e. element of $S_{n+m}$).
We already know that all semialgebraic sets form an o-minimal structure.

The most important property is the following :

### Theorem

*A definable set in an o-minimal structure has finitely many connected components and each component is definable.*

One can use this to prove for example that the graph of $\sin(x)$ is not definable (intersect with the line $y = 0$ for example). On the other hand, the restrcition to any bounded interval will be definable in an o-minimal structure.

# Some important o-minimal structures.

- All semialgebraic sets. Sometimes this structure is denoted by $\overline{\mathbb{R}}$.
- $\mathbb{R}_{exp}$. This structure includes all sets defined using the *real* exponential function : in this structure, the graph of the real exponential is definable.
  For example the subset of $\mathbb{R}^2$ : $\{(x, y) : y = x^2 e^{x^3} + x^5\}$ is definable in $\mathbb{R}_{exp}$.
  That $\mathbb{R}_{exp}$ is o-minimal is a theorem of Wilkie.
- $\mathbb{R}_{an}$. This structure contains all sets defined by 'restricted analytic functions'. A function $f : [-1, 1]^n \longrightarrow \mathbb{R}$ is a restrcited analytic function if it is a restriction of a real analytic function defined in a neighbourhood of $[-1, 1]^n$.
  This structure is o-minimal by a theorem of Van den Dries and (independently) Gabrielov.
- $\mathbb{R}_{an,exp}$. This structure includes both the real exponential functions and restricted analytic functions.

# Pila-Wilkie theorem

Let $H$ be the usual height of a rational number is defined as follows ($H(\frac{a}{b}) = \max(|a|, |b|)$ where $a$ and $b$ are coprime.)

For $(x_1, \ldots, x_n) \in \mathbb{Q}^n$, we define $H(x_1, \ldots, x_n) = \max(H(x_1), \ldots, H(x_n))$.

For $X \subset \mathbb{R}^n$ and $T \in \mathbb{R}_+$, define $X(\mathbb{Q}, T) = \{x \in X \cap \mathbb{Q}^n : H(x) \leq T\}$.
This is a finite set, let $N(X, T) = |X(\mathbb{Q}, T)|$.

PIla-Wilkie theorem concerns itself with estimating $N(X, T)$ for sets $X$ definable in an o-minimal structure.
Firstly, in general $N(X, T)$ can be large : if $X = \mathbb{R}^n$, then $N(X, T)$ grows like a polynomial of degree $n$ in $T$.
However, $\mathbb{R}^n$ is of course semialgebraic. Pila-Wilkie theorem says that if one removes from $X$ all positive dimensional semi-algebraic subsets, there are very few rational points up to height $T$ on what remains of $X$.

## Definition

*Let $X \subset \mathbb{R}^n$, the algebraic part $X^{alg}$ of $X$ is defined as the union of all infinite, connected semialgebraic subsets $Y \subset X$. The transcendental part $X^{tr}$ is $X \backslash X^{alg}$.*

We can now state Pila-Wilkie point counting theorem.

## Theorem (Pila-Wilkie)

*Let $X \subset \mathbb{R}^n$ be a set definable in some o-minimal structure. Let $\epsilon > 0$. There exists $C = C(X, \epsilon) > 0$ so that for $T \geq 1$, we have*

$$N(X^{tr}, T) \leq CT^{\epsilon}$$

Pila-Wilkie theorem extends to countng points defined over more general number fields.

For a subset $X \subset \mathbb{R}^n$ and $k \geq 1$, we define

$$N_k(X,T) = |\{x = (x_1,\ldots,x_n) \in X \cap \overline{\mathbb{Q}}^n : \deg(x_i) \leq k, H(x) \leq T\}|$$

We can now state the version of Pila-Wilkie theorem for number fields :

### Theorem (Pila-Wilkie, v2)

*Let $X \subset \mathbb{R}^n$ be a set definable in some o-minimal structure. Let $k \geq 1$. Let $\epsilon > 0$. There exists $C = C(X,k,\epsilon) > 0$ so that for $T \geq 1$, we have*

$$N_k(X^{tr},T) \leq CT^{\epsilon}$$

Further questions : Can one replace $T^{\epsilon}$ by a polynomial in $\log(T)$ ? Can we say something about the constant $C$ ? Is, for example. it is polynomial in $k$ ? These questions will be discussed in the lectures by Binyamini and Schmidt.

# Shimura varieties.

A Shimura datum is a pair $(G, X)$ where $G$ is a reductive group over $\mathbb{Q}$ and $X$ is a $G(\mathbb{R})$ orbit in $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ of an element $x_0$ satisfying certain conditions sufficient to ensure that $X$ is a finite union of Hermitian symmetric domains (it is usually not connected).
An example is $(\mathrm{GL}_2, \mathbb{H}^{\pm})$.

Let $(G, X)$ be a Shimura datum, $G$ is a reductive group over $\mathbb{Q}$, $K$ a compact open subgroup of $G(\mathbb{A}_f)$.
The Shimura variety associated to this data is :

$$Sh_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

It admits a canonical model over an explicitly described number field $E(G, X)$.

Let $X^+$ be a connected component of $X$ and $\Gamma = G(\mathbb{Q})^+ \cap K$.
Let $S = \Gamma \backslash X^+$. It is quasi-projective (Baily-Borel).
By the usual abuse of language we will still be calling $S$ a Shimura variety, and it admits a canonical model over a (well defined) abelian finite extension of $E(G, X)$.
**Special subvarieties** correspond to inclusions of Shimura subdata : $(G', X') \subset (G, X)$ and **special points** to $(T, x) \subset (G, X)$ where $T$ is a torus. The smallest such torus $T$ is called the **Mumford-Tate group** of $T$. Special points are defined over explicitly described number fields.

**Weakly special subvarieties** : those of the form
$S_1 \times \{x\} \subset S_1 \times S_2 \subset S$. They can be characterised as 'bi-algebraic' subvarieties.

# $\mathcal{A}_g$, the moduli space of principally polarised abelian varieties.

Let

$$\pi : \mathbb{H}_g = \{\tau \in M_g(\mathbb{C}), \tau = \tau^t, \mathrm{Im}(\tau) > 0\} \longrightarrow \mathcal{A}_g = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g,$$

be the uniformising map.
Then $\mathcal{A}_g$ is a moduli space for principally polarised Abelian varieties of dimension $g$,

$$\pi(\tau) = A_\tau = \mathbb{C}^g / (\mathbb{Z}^g \oplus \tau\mathbb{Z}^g).$$

and $\mathcal{A}_g$ is a quasi-projective algebraic variety defined over $\mathbb{Q}$.
Special points correspond to CM abelian varieties.
An abelian variety of dimension $g$ has CM if and only if $\mathrm{End}(A) \otimes \mathbb{Q}$ contains a commutative $\mathbb{Q}$-algebra of dimension 2g.
A Shimura variety $S$ is of **abelian type** if it admits a finite Shimura morphism $S' \longrightarrow S$ where $S'$ is a special subvariety of $\mathcal{A}_g$.

# Ingredients of the proof of AO

- Definability : $\pi\colon X \longrightarrow S$ is definable in $\mathbb{R}_{an,exp}$ when restricted to a suitable fundamental domain $\mathcal{F}$. (Klingler-Ullmo-Y)
- Functional transcendence (hyperbolic Ax-Lindemann theorem) : if $W$ is an algebraic subset of $X$, then $\pi(W)^{Zar}$ is weakly special. (Klingler-Ullmo-Y)
- Bounds on height of 'prespecial points' (Daw-Orr)
- Lower bounds for Galois orbits of special points (many authors).
- "geometric André-Oort" : assume $Z \subset S$ contains a Zariski dense set of $> 0$ dimensional weakly special subvarieties, then $Z = S_1 \times Z'$ where $S_1 \times S_2 \subset S$ is a special subvariety and $Z' \subset S_2$. (Ullmo, Richard-Ullmo)

# Lower bounds for Galois orbits

Let $s \in S$ be a special point corresponding to the inclusion of Shimura data $(T, x) \subset (G, X)$ where $T$ is the Mumford-Tate group of $x$. Assume $G$ adjoint, let $L$ be the splitting field of $T$. It is a CM field. Let $K_T^m$ be the maximal compact subgroup of $T(\mathbb{A}_f)$ and $K_T = K \cap T(\mathbb{A}_f)$.
Define the discriminant of $s$ as

$$d_s = |K_T^m / K_T||discr(L)|$$

**Lower bounds conjecture**

$$[\mathbb{Q}(s) : \mathbb{Q}] \gg d_s^{\delta}$$

where $\delta$ depends on $S$ only.
Ex. $E$ an elliptic curve with CM by $O_L$ ($L$ imaginary quadratic) then $[\mathbb{Q}(E) : \mathbb{Q}] \gg d_L^{1/4}$.
For $\mathcal{A}_g$ it was conjectured by Bas Edixhoven in 1999.

# Algebraic structure on $X$.

$X$ can be embedded as an open semialgebraic bounded subset in $\mathbb{C}^n$ where $n = \dim(X)$ (Harish-Chandra).
(Think of the open unit disc inside $\mathbb{C}$ - this is the Harish-Chandra realisation of the upper-half plane).
We then call a subset $W$ of $X$ algebraic if $W$ is the intersection of an algebraic subset of $\mathbb{C}^n$ and irreducible if it is an irreducible analytic component of such an intersection.
We have a transcendental map $\pi \colon X \longrightarrow S$ between two algebraic objects.
**Functional transcendence (Ax-Lindemann)** : if $W$ is an algebraic subset of $X$, then $\pi(W)^{Zar}$ is weakly special.
**Equivalently** : for $Z \subset S$ algebraic, maximal algebraic subsets of $\pi^{-1}(Z)$ are precisely components of preimages of weakly special subvarieties contained in $Z$.

This in particular implies a **bi-algebraic characterisation** of weakly special subvarieties : $Z \subset S$ (algebraic) is weakly special if and only if an analytic component of $\pi^{-1}(Z)$ is algebraic. In other words weakly specials are characterised as being bi-agebraic.

# Sketch of the proof of André-Oort.

Let $S$ be a Shimura variety, $Z \subset S$ a subvariety containing a Zariski dense set $\Sigma$ of special points. Consider $\pi\colon X \longrightarrow S$ and $\mathcal{F}$ a suitable fundamental domain.
Let $\widetilde{Z} := \mathcal{F} \cap \pi^{-1}Z$. This is a definable set by definability of the restriction of $\pi$ to $\mathcal{F}$.
For $s \in \Sigma$, let $x \in \mathcal{F}$ be such that $\pi(x) = s$ and let $d_s$ be the discriminant of $s$. Note : $d_s$ is unbounded as $s$ ranges through $\Sigma$.
By lower bounds for the Galois orbits : $[\mathbb{Q}(s) : \mathbb{Q})] \gg d_s^\delta$. Furthermore for any $x \in \mathcal{F}$ with $s = \pi(x) \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot s$, by Daw-Orr, $H(x) \ll d_s^\alpha$.
Thus $Z$ contains $\gg d_s^\delta$ points of uniformly bounded degree and height $\ll d_s^\alpha$. By Pila-Wilkie and Ax-Lindemann, through any $s$ with $d_s$ large enough, there passes a weakly special subvariety.
One concludes using 'geometric André-Oort'.

# Lower bounds for Galois orbits in the case of $\mathcal{A}_g$.

Let $A$ be an abelian variety of dimension $g$ with everywhere semistable reduction over a number field $L$. Let $h_F(A)$ be its Faltings height. Suppose that $A$ is simple and has CM by $O_E$ where $E$ is a CM field of degree $2\dim(A)$.
Andreatta, Goren, Howard, Madapusi-Pera and independently by Yuan and Zhang have proved an 'average Colmez formula' which implies that

$$\forall \epsilon > 0, h_F(A) \ll_\epsilon d_E^\epsilon$$

J. Tsimerman combined this with the following :
**Masser-Wustholz isogeny estimates :**
Let $A$ and $B$ be two abelian varieties of dimension $g$ over a number field $L$, isogeneous over $\mathbb{C}$. Let $N$ be the minimal degree of an isogeny between $A$ and $B$ over $\mathbb{C}$. Then

$$N \ll_g \max(h_F(A), [L : \mathbb{Q}])^{c_g}$$

where $c_g$ depends only on $g$.

... to deduce the "Edixhoven's conjecture" :

$$[\mathbb{Q}(A) : \mathbb{Q}] \gg d_E^{\delta_g}$$

This implies A-O for all Shimura varieties of abelian type.

The problem with generalising this is that there is no evident (even conjectural analogue) of the Masser-Wustholz theorem for general Shimura varieties.
Harry Schmidt had an idea of using point counting to approach this kind of problem.
This requires a significantly stronger version of Pila-Wilkie theorem, obtained by G. Biniyamini.

# Biniyamini's point counting

Let $\pi\colon X \longrightarrow S$ be the uniformisation of a Shimura variety. Let $E$ be its number field of definition. Consider $X \times S \subset \mathbb{C}^n \times S$ and let $h$ be some (logarithmic !) Weil height function on $\mathbb{C}^n \times X$.
Let $Z_S = \{(x,s) : x \in \mathcal{F}, s = \pi(x)\} \subset X \times S$.

$$Z_S(f,h) = \{(x,s) \in Z_S : [E(x,s) : E] \leq f, h(x,s) \leq h\}$$

**Biniyamini's theorem**

$$|Z_S(f,h)| \ll f^A h^B$$

where $A$ and $B$ depend on $S$ only.

# Lower bounds for Galois orbits.

Let $x$ be a special point of $\mathcal{F}$ and $s = \pi(x)$.
Consider $S(s)$, the smallest zero dimensional Shimura variety containing $s$.
Its size is the class group of $T$ which is bounded below by $d_s^{\alpha}$. All elements of $S(s)$ are defined over a field of degree bounded by $f$.
**Conjecture - Biniyamini, Schmidt, Y** With respect to some Weil height $h$ on $S$,

$$h(s) \ll_{\epsilon} d_x^{\epsilon}$$

**Theorem of Daw-Orr**

$$H(x) \ll \log(d_x)$$

where $C$ is some constant.
It follows that $h(x,s) \ll d_x^{\epsilon}$ for all $(x,s)$ in $\mathcal{F} \times S$ with $s = \pi(x) \in S(s)$.

**Theorem (Ullmo-Y, Tsimerman)**

$$|S(s)| \gg d_x^{\delta_g}$$

The point $x$ is algebraic and its degree is uniformly bounded. We therefore have :

$$d_x^{\delta_g} \ll f^A d_x^{\epsilon}$$

This implies a lower bound for $f$ of required type.
**Theorem (Pila-Shankar- Tsimerman, Esnault, Groechenig)**
The height conjecture is true.