Harmonic analysis of translation-invariant valuations and geometric inequalities

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Integral and Metric Geometry

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Alesker theory

Valuations on convex bodies

$\mathcal{K} = \{$convex bodies in $\mathbb{R}^n\}$.

$\phi\colon \mathcal{K} \to \mathbb{C}$ is a **valuation** if $\phi(K \cup L) = \phi(K) + \phi(L) - \phi(K \cap L)$.

$\mathrm{Val} = \{$continuous, translation-invariant valuations on $\mathcal{K}\}$.

Theorem (McMullen 1977)

$$\mathrm{Val} = \bigoplus_{k=0}^{n} \mathrm{Val}_k\,.$$

Example

$$K \mapsto V(K[k], C_{k+1}, \dots, C_n) \in \mathrm{Val}_k\,.$$

Proposition

$$\mathrm{Val}_0 = \mathbb{C}.$$

Theorem (Hadwiger 1957)

$$\mathrm{Val}_n = \mathbb{C} \cdot \mathrm{vol}_n\,.$$

Irreducibility theorem

The $GL(n)$ action on Val ($g \cdot \phi = \phi \circ g^{-1}$) preserves $\mathrm{Val}_k = \mathrm{Val}_k^{even} \oplus \mathrm{Val}_k^{odd}$.

Theorem (Alesker 2001)
The $GL(n)$ representations $\mathrm{Val}_k^{\varepsilon}$ are irreducible.

Corollary (McMullen's conjecture)
Mixed volumes span a dense subset of Val.

Corollary (Smooth valuations)
$\phi \in \mathrm{Val}^{\infty}$ if and only if

$$\phi(K) = c\,\mathrm{vol}_n(K) + \int_{N(K)} \omega$$

for some $c \in \mathbb{C}$ and $\omega \in \Omega^{n-1}(\mathbb{R}^n \times S^{n-1})^{tr}$.

Product of valuations

Theorem (Alesker 2004)
Val^∞ is a commutative graded algebra.

Theorem (Alesker 2004, Bernig 2008)
The product pairing $\mathrm{Val}_k^\infty \times \mathrm{Val}_{n-k}^\infty \to \mathrm{Val}_n^\infty \cong \mathbb{C}$ is non-degenerate.

Theorem (Bernig–Fu 2006)
There exists a unique bilinear, continuous product on Val^∞ s.t.

$$V\big(K_1, \ldots, K_k, \cdot[n-k]\big) * V\big(L_1, \ldots, L_l, \cdot[n-l]\big)$$
$$= c_{n,k,l} V\big(K_1, \ldots, K_k, L_1, \ldots, L_l, \cdot[n-k-l]\big).$$

Fourier transform

Theorem (Alesker 2011)
There is an isomorphism $\mathbb{F} : \mathrm{Val}^\infty \to \mathrm{Val}^\infty$ s.t.

(a) $\mathbb{F}$ commutes with the action of $SO(n)$,

(b) $\mathbb{F}\,\mathrm{Val}_k^\infty = \mathrm{Val}_{n-k}^\infty$,

(c) $\mathbb{F}(\phi \cdot \psi) = \mathbb{F}\phi * \mathbb{F}\psi$,

(d) $(\mathbb{F}^2\phi)(K) = \phi(-K)$.

Conjecture (K. 2021)

Let $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $\psi_i = V\big(C_i, \cdot[n-1]\big) \in \mathrm{Val}_{n-1}^\infty$. Then the following holds:

(a) *Hard Lefschetz theorem.* The map $\mathrm{Val}_{n-k}^\infty \to \mathrm{Val}_k^\infty$ given by

$$\phi \mapsto \phi * (\psi_1 * \cdots * \psi_{n-2k})$$

is an isomorphism.

(b) *Hodge–Riemann relations.* If $\phi \in \mathrm{Val}_{n-k}^\infty$ satisfies

$$\phi * (\psi_1 * \cdots * \psi_{n-2k}) * \psi_{n-2k+1} = 0,$$

then

$$(-1)^k \phi * \overline{\phi} * (\psi_1 * \cdots * \psi_{n-2k}) \geq 0.$$

Proven in special cases:

Alesker 2003, Bernig–Bröcker 2008,

K. 2021, K.–Wannerer 2021 (indep. Alesker), K.–Wannerer 2022.

Harmonic analysis

ABS theorem

Assume $n = 2l$.

Highest weights for $SO(2l)$: $\left\{(\lambda_1, \ldots, \lambda_l) \in \mathbb{Z}^l \mid \lambda_1 \geq \cdots \lambda_{l-1} \geq |\lambda_l|\right\}$.

Theorem (Alesker–Bernig–Schuster 2011)
Val_r and Val_{n-r} decompose into these irreducible representations of $SO(2l)$:

$$(0, \ldots, 0);$$

$$(m, \underbrace{2, \ldots, 2}_{k-1}, 0, \ldots, 0), \quad m \geq 2, \quad k \leq r;$$

$$(m, 2, \ldots, 2, -2), \qquad m \geq 2 \quad \text{(only if } r = l\text{)}.$$

Moreover, each occurs with multiplicity one.

Double forms

Complex coordinates on $\mathbb{R}^n \times \mathbb{R}^n$:

$$z_j = \frac{1}{\sqrt{2}}\left(x_{2j-1} + \sqrt{-1}x_{2j}\right), \qquad \zeta_j = \frac{1}{\sqrt{2}}\left(\xi_{2j-1} + \sqrt{-1}\xi_{2j}\right),$$
$$z_{\bar{j}} = \frac{1}{\sqrt{2}}\left(x_{2j-1} - \sqrt{-1}x_{2j}\right), \qquad \zeta_{\bar{j}} = \frac{1}{\sqrt{2}}\left(\xi_{2j-1} - \sqrt{-1}\xi_{2j}\right).$$

Double forms: $\Omega^*(\mathbb{R}^n \times S^{n-1}) \otimes \bigwedge_{\mathbb{C}}(\mathbb{R}^n)^*$.

For $I \subset \{1, \bar{1}, \dots, l, \bar{l}\}$ we define

$$\zeta_I = \sum_{i \in I} \zeta_i \otimes dz_i, \quad d\zeta_I = \sum_{i \in I} d\zeta_i \otimes dz_i, \quad dz_I = \sum_{i \in I} dz_i \otimes dz_i.$$

We will consider

$$K = \{1, 2, \dots, k\} \quad \text{and} \quad J = K^c.$$

Denote

$$\omega_{r,k} \otimes (dz_1 dz_{\bar{1}} \cdots dz_l dz_{\bar{l}}) = \zeta_J (d\zeta_J)^{n-r-1} (dz_J)^{r-k} (\overline{dz_K})^k$$

and

$$\omega_{r,k,m} = \zeta_{\bar{1}}^{m-2} \omega_{r,k}.$$

Theorem (K.–Wannerer 2022)
The valuation

$$\phi_{r,k,m}(K) = c_{n,r,k,m} \int_{N(K)} \omega_{r,k,m}$$

is a non-trivial highest weight vector in Val_r of weight

$$(m, \underbrace{2, \ldots, 2}_{k-1}, 0, \ldots, 0).$$

Theorem (K.–Wannerer 2022)
(a) Alesker–Poincaré pairing.

$$\phi_{r,k,m} * \overline{\phi_{n-r,k,m}} = (-1)^k a_{n,r,k,m}.$$

(b) *Fourier transform.*

$$\mathbb{F}\phi_{r,k,m} = (-1)^{k-1}(\sqrt{-1})^m \phi_{n-r,k,m}.$$

(c) *Lefschetz operator.*

$$V\big(B, \cdot[n-1]\big) * \phi_{r,k,m} = \begin{cases} b_{n,r,k,m}\,\phi_{r-1,k,m} & \text{if } k < r, \\ 0 & \text{if } k = r. \end{cases}$$

Theorem (K.–Wannerer 2022)

If $\phi \in \mathrm{Val}_{n-k}^{\infty}$ satisfies $\phi * V\big(B[n-2k+1], \cdot[2k-1]\big) = 0$, then

$$(-1)^k \phi * \overline{\phi} * V\big(B[n-2k], \cdot[2k]\big) \geq 0.$$

Corollary (Van Handel)

(HR) holds for $n \leq 4$.

Corollary (Alesker 2021)

If $K_1, \dots, K_N, L_1, \dots, L_N, M \in \mathcal{K}_+^{\infty}$ and $\alpha_1, \dots, \alpha_N \in \mathbb{R}$ satisfy

$$\sum_{i=1}^{N} \alpha_i V(K_i, L_i, M, \cdot) = 0,$$

then

$$0 \leq \sum_{i,j=1}^{N} \alpha_i \alpha_j V(K_i, K_j, L_i, L_j).$$

Thank you!

Definition (Alesker 2011)
Let $\phi \in \mathrm{Val}(\mathbb{R}^n)$ and $K \in \mathcal{K}(\mathbb{R}^{n-1})$. We define
(a) *pullback* along the inclusion $\iota : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \oplus \mathbb{R}$: $\quad (\iota^*\phi)(K) = \phi(\iota K)$;
(b) *pushforward* along the projection $\pi : \mathbb{R}^{n-1} \oplus \mathbb{R} \to \mathbb{R}^{n-1}$:

$$(\pi_*\phi)(K) = \left.\frac{d}{dt}\right|_{t=0} \phi\big(K + [0,t]\big).$$

Theorem (Alesker 2011)
$\mathbb{F} \circ \iota^* = \pi_* \circ \mathbb{F}$.

Theorem (K.–Wannerer 2022)

$$\iota^*\phi_{r,k,m} = \begin{cases} \phi^{(n-1)}_{r,k,m} & \text{if } r < n-1 \text{ and } k < n-r, \\ \frac{1}{2}\phi^{(n-1)}_{k,k-1,m} & \text{if } k = \frac{n}{2}. \end{cases}$$

$$\pi_*\phi_{r,k,m} = \begin{cases} \phi^{(n-1)}_{r-1,k,m} & \text{if } k < r, \\ -\frac{1}{2}\phi^{(n-1)}_{k-1,k-1,m} & \text{if } k = \frac{n}{2}. \end{cases}$$