Advances in Mixed Characteristic Commutative Algebra & Geometric Connections

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1 Overview of the Field

Commutative algebra is a classical subject that studies properties of commutative rings. Commutative algebra is intimately tied to algebraic geometry in several ways. First, local properties of algebraic varieties (i.e., singularities) can be and are studied by studying their rings of functions. Alternately, projective varieties can be studied by considering their graded homogeneous coordinate rings. This latter perspective can be viewed as studying properties of the cone over the projective variety, and hence also can be viewed as studying singularities.

When the commutative ring being studied contains a field, there are very well developed and well established tools to study the associated singularities. In characteristic 0, one can use geometric techniques: the theory of differential operators, resolution of singularities, and vanishing theorems from algebraic geometry. In characteristic $p > 0$, the Frobenius map and tight closure theory play a central rule. Unfortunately, most of these techniques do not translate well to the mixed characteristic case (i.e., when the ring does not contain a field, for instance the integers or the p-adic integers), and many fundamental questions in commutative algebra have remained open in the mixed characteristic setting.

In recent years, there have been several extremely successful attempts to attack some of these mixed characteristic problems. One of the new inputs here is the work of Scholze, and applications to commutative algebra by André, Bhatt, Gabber and others, including many of the participants. These recent advances in mixed characteristic commutative algebra open the door to a unified picture across all the characteristics and lead to a wide variety of new applications. Of a particular note, the results produced in mixed characteristic, and their proofs, often closely mirror results and techniques in positive characteristic. The goal of this workshop is to bring experts and young researchers together to work in this new and exciting area.

2 Recent Developments and Open Problems

There has been spectacular progress in mixed characteristic in the past 8 years. Notably, a real breakthrough happened in 2016 when Yves André proved the Direct Summand Conjecture (now André's Theorem), namely that if A is a regular ring and $A \subseteq R$ is a finite ring extension, then the map $A \to R$ splits as a map of R-modules. André and Gabber also proved the existence of (weakly functorial) big Cohen-Macaulay algebras, a generalization of the direct summand conjecture. These results imply most of the so called "Homological

Conjectures" in commutative algebra pioneered by Peskine–Szpiro, Hochster and Roberts. The equal characteristic case of most of these conjectures follow from work of Hochster more than 40 years ago. Some previous notable progress in this field in the mixed characteristic case are due to Roberts and Heitmann.

More recently Bhatt proved that the completion of the absolute integral closure of a complete local domain is itself big Cohen-Macaulay. This is the mixed characteristic extension of a celebrated result of Hochster–Huneke in positive characteristic. Bhatt's result played a central rule in recent progress in the minimal model program for threefolds in mixed characteristic. Thanks to this theorem, now there has been substantial work in understanding splinters (that is, rings that splits off from all their finite extensions) in both equal and mixed characteristics. In another direction, important work of De Stefani-Grifo-Jeffries generalized the classical Zariski-Nagata theorem on symbolic powers to polynomial rings over the integers or p-adic numbers, using the theory of p-derivations developed by Buium and Joyal. Again using p-derivations, Hochster-Jeffries gave a Jacobian type criterion for regularity in mixed characteristic. These works have been applied by many of the speakers and organizers to the study of local cohomology modules, symbolic powers, and birational geometry and singularities in mixed characteristic.

Despite all of this exciting progress, there are still many important questions to answer. This is reflected in the talks of many speakers who brought up open problems. Moreover, to actively facilitate further development in the field, we held an open problem session on the second day of the conference. Jack Jeffries and Elisa Grifo talked about open questions on p-derivations and symbolic powers in positive and mixed characteristic, Jean Chan discussed some open problems in Hilbert-Kunz multiplicity, and Linquan Ma discussed an open problem (essentially due to Huneke) on multiplicities with connections to the homological conjectures.

3 Presentation Highlights

1. **Quasi-F-splittings**
   Jakub Witaszek (University of Michigan)

   Jakub Witaszek talked about his joint work with Tatsuro Kawakami, Hiromu Tanaka, Teppei Takamatsu, Fuetaro Yobuko, and Shou Yoshikawa on Quasi-F-splitting, which is a new notion in characteristic $p > 0$ but is defined via lifting to Witt vectors. This is a natural generalization of F-splitting and has surprising applications to liftability problems in characteristic $p > 0$ birational geometry. Many related open questions are discussed as well.

2. **Test ideals in all characteristics via closure-interior duality**
   Rebecca R.G. (George Mason University)

   A number of the tools used in studying singularities in commutative algebra have come from the study of tight closure and its test ideal in rings of equal characteristic. In replicating these results in mixed characteristic, it has been useful to find test ideal-like structures for a variety of new closure operations. Then the speaker discussed hot apply the closure-interior duality laid out by the speaker and Neil Epstein to describe properties shared by many test ideals, and use these to give results on particular test ideals.

3. **Untilting Line Bundles on Perfectoid Spaces**
   Gabriel Dorfsman-Hopkins (University of California, Berkeley)

   Let be a perfectoid space with tilt . IT was build a natural map where the (inverse) limit is taken over the -power map, and show that is an isomorphism if is a perfectoid ring. As a consequence we obtain a characterization of when the Picard groups of and agree in terms of the -divisibility of . The main technical ingredient is the vanishing of certain higher derived limits of the unit group , whence the main result follows from the Grothendieck spectral sequence.

4. **Hilbert-Kunz multiplicity of powers of an ideal**
   Kriti Goel (University of Utah)

   This talks discussed suitable conditions under which the asymptotic limit of the Hilbert-Samuel coefficients of the Frobenius powers of an -primary ideal exists in a Noetherian local ring with prime characteristic This, in turn, gives an expression of the Hilbert-Kunz multiplicity of powers of the ideal. It was also proved that for a face ring of a simplicial complex and an ideal generated by pure powers of the variables, the generalized Hilbert-Kunz function is a polynomial for all and also give an expression of the generalized Hilbert-Kunz multiplicity of powers of in terms of Hilbert-Samuel multiplicity. The talk concluded by giving a counter-example to a conjecture proposed by I. Smirnov which connects the stability of an ideal with the asymptotic limit of the first Hilbert coefficient of the Frobenius power of the ideal.

5. **Symbolic powers in mixed characteristic**
   Eloísa Grifo (University of Nebraska – Lincoln)

   In a polynomial ring over a perfect field, a classical theorem of Zariski and Nagata says that the symbolic powers of a radical ideal I – which are defined by taking the minimal primary components of I^n – coincide with its differential powers. This description fails in mixed characteristic: one needs to also consider differential-like operators that decrease p-adic order for a prime integer p. The solution turns out to be Buium and Joyal's p-derivations. This talk was a joint work of the speaker with Alessandro De Stefani and Jack Jeffries.

6. **A uniform Chevalley theorem for direct summands in mixed characteristic**
   Alessandro De Stefani (Università degli Studi di Genova)

   Let R be a graded direct summand of a positively graded polynomial ring over the p-adic integers. In the talk it was exhibited an explicit constant D such that, for any positive integer n and any homogeneous prime ideal Q of R, the n-th symbolic power of Q is contained in the n-th power of the homogeneous maximal ideal $(p)R + R_+$. The strategy relies on a Zariski-Nagata type of theorem that works in mixed characteristic, together with the introduction of a new type of differential powers which do not require the existence of a p-derivation on R. The talk was based on joint work with E. Grifo and J. Jeffries.

7. **Integral p-adic cohomology for open and singular varieties**
   Veronika Ertl (University Regensburg)

8. **Splinter rings and Global +-regularity**
   Kevin Tucker (University of Illinois Chicago)

   A Noetherian ring is a splinter if it is a direct summand of every finite cover. In this talk,the speaker discussed some recent work on splinter rings in both positive and mixed characteristics. In particular, inspired by the result of Bhatt on the Cohen-Macaulayness of the absolute integral closure, the speaker described a global notion of splinter in the mixed characteristic setting called global +-regularity with applications to birational geometry in mixed characteristic. This can be seen as a generalization of the theory of globally F-regular pairs from positive to mixed characteristic. This talk was based on the joint work arXiv:2012.15801 with Bhargav Bhatt, Linquan Ma, Zsolt Patakfalvi, Karl Schwede, Joe Waldron, and Jakub Witaszek.

9. **Nash blowup of toric surfaces in positive characteristic**
   Daniel Duarte (CONACyT-Universidad Autónoma de Zacatecas)

   The Nash blowup was almost exclusively studied in characteristic zero due to example by Nobile. Very recently, it was shown that this blowup is still significant for normal varieties in prime characteristic. The speaker presented a combinatorial description of Nash blowups of toric varieties in positive characteristic. Then, it was shown that normalized Nash blowups solve the singularities of normal toric surfaces over fields of positive characteristic.

10. **A Jacobian Criterion in Mixed Characterstic**
    Jack Jeffries (University of Nebraska – Lincoln)

    The classical Jacobian criterion is an important tool for finding singular points on a variety over a (perfect) field. How can we find the singular locus over the p-adics or over the integers? In this talk, it was discussed a new analogue of the Jacobian criterion that gives a simple description of the singular locus in this setting. This criterion uses a curious notion of differentiation by a prime number called p-derivations. It was also discussed an extension of the theory of Kahler differentials to this mixed characteristic setting. This was based on joint work with Melvin Hochster.

11. **Arithmetic deformations of F-singularities**
Kenta Sato (Kyushu University)

Recently, using perfectoid techniques, Ma and Schwede developed a new theory of singularities in mixed characteristic. As an application of this theory, they proved that a Q-Gorenstein affine domain over a field of characteristic zero has log terminal singularities if its mod p reduction is F-regular for one single prime p. In this talk, as a new application of their theory, I will discuss the analog of their result for log canonical singularities. This talk is based on joint work with Shunsuke Takagi.

12. **Test Ideals in Mixed Characteristic**
Alicia Lamarche (University of Utah)

The speaker discussed recent joint work with Christopher Hacon and Karl Schwede in which they define a notion of test ideals for rings of finite type over a complete local Noetherian ring that commutes with localization. It was also discussed applications of our definition.

13. **Recent advances in understanding splinters**
Rankeya Datta (Michigan State University)

Inspired by the direct summand theorem, it is natural to ask what rings satisfy the property that they split off from all their module-finite extensions. These rings are called splinters and Rankeya Datta give an talk surveying recent progress in understanding these rings. For example, it is shown that under mild assumptions on the ring R, the splinter loci (that is, those $P \in Spec(R)$ such that R_P is a splinter) is a Zaraski open subset of $Spec(R)$ in positive characteristic. The speaker also highlighted on a recent result of Shiji Lyu that if $R \to S$ is a regular homomorphism in positive characteristic and R is a splinter, then S is a splinter (the proof, surprisingly, utilizes ultraproduct and the openness result mentioned earlier).

14. **Rational singularities 2.0**
Sandor Kovacs (University of Washington)

Sandor Kovacs talked about his updated result towards an understanding of rational singularities from a characteristic-free point of view. The main result is that if one has a projective birational morphism f: $Y \to X$ of excellent schemes such that Y is Cohen-Macaulay and normal and X has pseudo-rational singularities, then $Rf_*O_Y = O_X$ and $Rf_*\omega_Y = \omega_X$. It is highlighted that this result holds in arbitrary characteristic and the normal assumption on Y is necessary (otherwise the conclusion is false and an elementary example is presented). It is then natural to ask that for a given excellent normal scheme X, whether one can finds a normal Macaulayfication, that is, a normal Cohen-Macaulay scheme Y such that $Y \to X$ is projective and birational – if so then the result can be used to give a characteristic free definition of rational singularities (and it coincides with pseudo-rational singularities). It is known that Macaulayfication always exists in arbitrary characteristic, but it is not clear that we can find a Macaulayficaition that is also normal beyond the case that the non-CM locus has dimension at most one.

15. **Fedder type criteria for quasi-Frobenius-splitting**
Teppei Takamatsu (Kyoto University)

In algebraic geometry of positive characteristic, singularities defined by the Frobenius map, including the notion of Frobenius-splitting, have played a crucial role. Moreover, there are powerful criteria, so-called Fedder's criteria, to confirm such properties. Yobuko recently introduced the notion of quasi-F-splitting and F-split heights, which generalize and quantify the notion of Frobenius-splitting, and proved that F-split heights coincide with Artin-Mazur heights for Calabi-Yau varieties. This notion is defined for purely positive characteristic varieties, but the ring of Witt vectors, which is a mixed characteristic object, makes an essential role in the definition. In this talk, The speakr gave a generalization of

Fedder's criteria to quasi-Frobenius-splitting, and introduce examples and applications of such criteria. This talk is based on a joint paper with Tatsuro Kawakami and Shou Yoshikawa.

16. **Toward improving Lech's inequality**
Ilya Smirnov (BCAM-Basque Center for Applied Mathematics)

 Ilya Smirnov talked about his joint work with Craig Huneke, Linquan Ma and Pham Hung Quy on Lech's inequality. The focus was to explore conditions on the ring that can make the classical Lech's inequality sharp. It was shown that Lech's inequlality can be improved uniformly under mild assumptions on the ring in all characteristics. Moreover, it is expected that under stronger assumptions, Lech's inequality can be vastly sharpend for ideals of sufficiently large colength, this last question remains wide open in characteristic 0 and mixed characteristic.

17. **Generalized valuations and idempotization of algebraic varieties**
Cristhian Garay López (Centro de Investigación en Matemáticas)

 Classical valuation theory has proved to be a valuable tool in number theory, algebraic geometry and singularity theory. Krull valuations defined on rings induce topologies on them, since these are maps taking values in totally ordered abelian groups, which carry an intrinsic topology themselves. One then uses these ring topologies to enrich algebraic varieties with new points coming from valuations. In this talk the speaker considers valuations that might take values in certain lattice-ordered abelian monoids known as lattice-ordered semirings. For example, it has been used to construct geometric objects out of (commutative) rings endowed with a valuative topology. As an application of this set of ideas, it was shown how to associate an idempotent version of the structure sheaf of an algebraic variety over an arbitrary field, which behaves particularly well with respect to idempotization of closed subschemes. This was based on a joint work with S. Falkensteiner, M. Haiech, M. P. Noordman and L. Bossinger.

4 Scientific Progress Made

During the meeting, numerous results were disseminated by various speakers. These will help inspire future research. The open problem session, and open problems highlighted by the speakers, will also hopefully inspire future research. Additionally, there was a great deal of informal mathematical discussion at the conference. Indeed, for many participants this conference was the first chance people had for in-person mathematical discussions, outside their home institutions, in the past two years. We therefore expect that this conference will lead to new collaborations in the future.

5 Outcome of the Meeting

This hybrid meeting was a success.

References