The classifying space of the G-cobordism category

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Plan of the talk

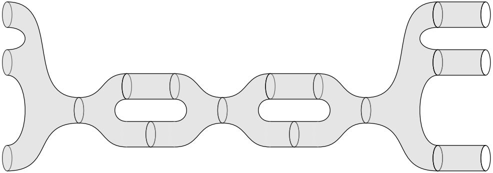
- Cobordism category and Tillmann's method
- Define the G-cobordism category $\mathscr{S}^G$
- Correspondence $\pi_0(\mathscr{S}^G) \cong G/[G,G]$
- Show the splitting short exact sequence

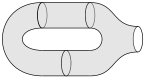
$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(\mathscr{S}^G) \longrightarrow \Omega_2^{SO}(G) \longrightarrow 0$$

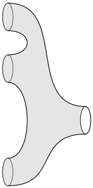
- The classifying space $B\mathscr{S}^G \simeq G/[G,G] \times S^1 \times X^G$
- Applications

Cobordism Category $\mathscr{S}$

- Objects: Finite disjoint union of circles.
- Morphisms: Two dimensional cobordism,

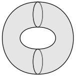




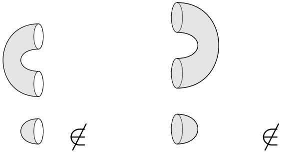


Tillmann's method

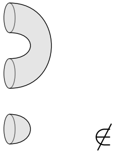
- $\mathscr{S}_0$ is the full subcategory of $\mathscr{S}$ with only one object given by the empty 1-dimensional manifold.



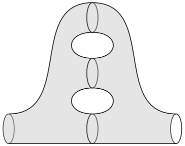
- $\mathscr{S}_{>0}$ is the subcategory of $\mathscr{S}$ with the same objects of $\mathscr{S}$ except for the empty manifold and where each connected component of every morphism has non empty incoming boundary and non empty outgoing boundary.



- $\mathscr{S}_b$ is the subcategory of $\mathscr{S}$ with the same objects of $\mathscr{S}$ and where each connected component of every morphism has non empty outgoing boundary.



- $\mathscr{S}_1$ is the full subcategory of $\mathscr{S}_{>0}$ with only one object, the circle.

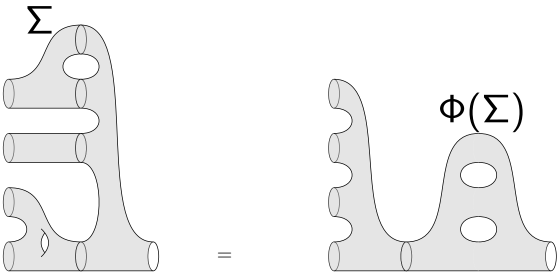


From this picture what did you see?

- There are isomorphisms $\mathscr{S}_0 \cong \mathbb{N}^\infty$ and $\mathscr{S}_1 \cong \mathbb{N}$ ($B\mathscr{S}_0$ is the infinite dimensional torus T^∞ and $B\mathscr{S}_1 \simeq S^1$).
- It is defined the functor $\Phi : \mathscr{S}_{>0} \longrightarrow \mathscr{S}_1$, which is constant map in objects and each morphism Σ with n incoming circles, c connected components, genus g and m outgoing circles ($\Sigma : n \to m$), is mapped to

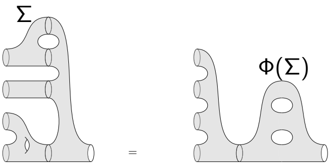
$$\Phi(\Sigma) = \frac{1}{2}(m - n - \chi(\Sigma)) = g + m - c$$

Example for $\Sigma : 4 \to 4$:



- We have the adjoint functors $\Phi : \mathscr{S}_{>0} \to \mathscr{S}_1 \quad \dashv \quad i : \mathscr{S}_1 \hookrightarrow \mathscr{S}_{>0}$ which is

$$\begin{array}{ccc} n & \xrightarrow{p_n} & 1 \\ {\scriptstyle \Sigma}\downarrow & & \downarrow{\scriptstyle \Phi(\Sigma)} \\ m & \xrightarrow{p_m} & 1 \end{array}$$



$\Phi \circ i = \mathrm{id}_{\mathscr{S}_1}$ and $i \circ \Phi \simeq \mathrm{id}_{\mathscr{S}_{>0}}$

- We can extend to Φ to the whole category $\mathscr{S}$ by negative values, $\Phi' : \mathscr{S} \to \mathbb{Z}$. The composition $\mathbb{N} \cong \mathscr{S}_1 \hookrightarrow \mathscr{S} \xrightarrow{\Phi'} \mathbb{Z}$ is a homotopy equivalence.

Theorem (Tillmann)

$B\mathscr{S}_0 \simeq T^\infty$, $B\mathscr{S}_1, B\mathscr{S}_{>0}, B\mathscr{S}_b \simeq S^1$ and $B\mathscr{S} \simeq X \times S^1$ where X is a simply connected infinite loop space.

- Tillmann (1996), The classifying space of the 1+1 cobordism category $\mathscr{S}$ (Crelle)
- Galatius-Madsen-Tillmann-Weiß(2009) The homotopy type of the cobordism category Cob_d (Act.Math.)

Theorem (T)

- $\pi_0(B\mathscr{S}) = 0$ $(\Omega_1^{SO} = 0)$,
- $\pi_1(B\mathscr{S}) = \mathbb{Z}$, *and*
- $B\mathscr{S} \simeq S^1 \times X$, $\pi_1(X) = 0$.

Theorem (GMTW)

- $\pi_0(B\,\mathrm{Cob}_2) = 0$,
- $\pi_1(B\,\mathrm{Cob}_2) = \mathbb{Z}$,
- $B\,\mathrm{Cob}_2 \simeq \Omega^{\infty-1} MTO(2)$,

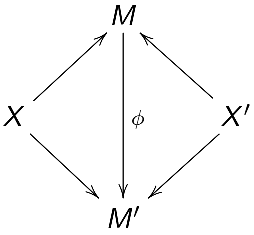
where $MTO(d)$ certain Thom spectrum.

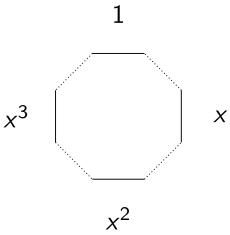
$$\Omega_{d-1} = \pi_0(\mathrm{Cob}_d) = \pi_0(\Omega^{\infty-1} MTO(d)) = \pi_{d-1} MO$$

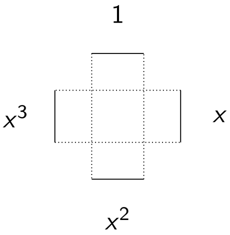
Thom-Pontriaguin

The G-cobordism category $\mathscr{S}^G$

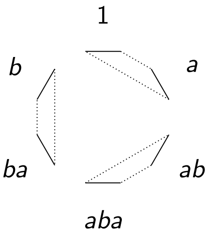
- **Objects:** Finite sequence $(x_1, x_2, \cdots, x_n)$, with $x_i \in G \sqcup \{0\}$ representing the disjoint union of principal G-bundles over the circle for $x_i \in G$ and the empty G-bundle for $x_i = 0$.
- **Morphisms:** Free cobordism classes of principal G-bundles. Recall a cobordism from X to X' manifolds of the same dimension, is a manifold M with $\partial M = X \sqcup -X'$. Two cobordisms (X, M, X') and (X, M', X') represent the same class if there is a diffeomorphism $\phi : M \to M'$ with the commutative diagram

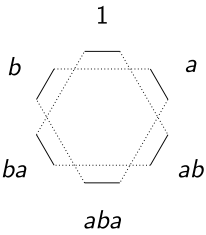




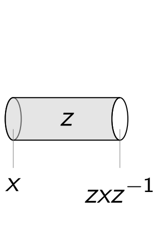


$$x, x^2 \in \mathbb{Z}/\mathbb{Z}_4 = \{1, x, x^2, x^3\}.$$

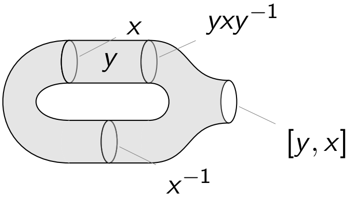




$$a, ab \in S_3 = \langle a, b : a^2 = b^2 = (ab)^3 = 1 \rangle.$$



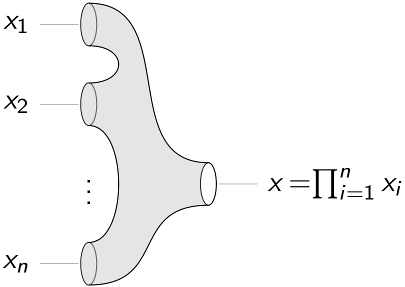
G-cobordisms over the cylinder, the pair of pants and the disc.



G-cobordism over a handlebody with exit $[y, x]$.

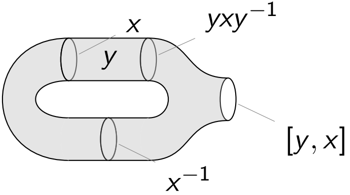
The connected components $\pi_0(\mathscr{S}^G)$

- The G-cobordism over disc implies that we can reduce to the subcategory with sequences $(x_1, \cdots, x_n)$ with $x_i \neq 0$.
- We use the G-cobordism over a pair of pants with multiple legs



- Then we restrict to the subcategory with objects of the form $x \in G$.

- Disregarding any components of G-cobordisms over closed surfaces and due to the G-cobordisms over handlebodies



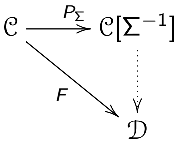
- Thus two elements $x, y \in G$ are connected in $\mathscr{S}^G$ if and only if they differ by an element in $[G, G]$.
- Consequently,

$$\pi_0(B\mathscr{S}^G) = G/[G, G]$$

- Since $\mathscr{S}^G$ is a symmetric monoidal category, the space $B\mathscr{S}^G$ is a grouplike with abelian fundamental group and hence any two connected components have the same homotopy type.

The fundamental group $\pi_1(\mathscr{S}^G)$

- Let $\mathcal{C}$ be a category (connected) and a subset of morphism Σ closed, we associate the localization $\mathcal{C}[\Sigma^{-1}]$ and functor $P_\Sigma : \mathcal{C} \longrightarrow \mathcal{C}[\Sigma^{-1}]$:
  - $P_\Sigma(f)$ is invertible for $f \in \Sigma$,
  - if $F : \mathcal{C} \longrightarrow \mathcal{D}$ inverts $F(f)$, for $f \in \Sigma$,



- $\text{Fun}(\pi_1(\mathcal{C}), \text{Set}) \overset{\sim}{\leftarrow} \text{Cov}(\mathcal{C}) \overset{\sim}{\rightarrow} \text{Fun}_{inv}(\mathcal{C}, \text{Set}) \cong \text{Fun}(\mathcal{C}[\mathcal{C}^{-1}], \text{Set}) \overset{\sim}{\rightarrow} \text{Fun}(\mathcal{C}[\mathcal{C}^{-1}]_x, \text{Set})$

Theorem (Quillen)

There is an isomorphism $\pi_1(\mathcal{C}, x) \cong \text{Hom}_{\mathcal{C}[\mathcal{C}^{-1}]}(x, x)$.

Left calculus of fractions

Definition (Localizing set)

For $\mathcal{C}$ a subset of morphisms Σ is localizing if we have:

1. Σ contains the identities and Σ is closed under composition.
2. $f \in \mathcal{C}$, $s \in \Sigma$ there exist $g \in \mathcal{C}$ and $t \in \Sigma$ with

$$\begin{array}{ccc} w & \xrightarrow{g} & z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ x & \xrightarrow[f]{} & y \end{array}$$

3. $f, g : x \to y$ two morphisms, if $s \circ f = s \circ g$ for $s \in \Sigma$, if and only if, $f \circ t = g \circ t$ for $t \in \Sigma$.

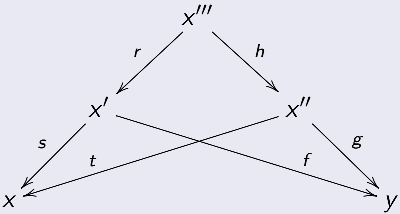
Theorem (Grabiel-Zisman)

Let Σ be a localizing subset of morphisms in $\mathcal{C}$. The category $\mathcal{C}[\Sigma^{-1}]$ can be described:
The objects of $\mathcal{C}[\Sigma^{-1}]$ are the same of $\mathcal{C}$. One morphism $x \longrightarrow y$ in $\mathcal{C}[\Sigma^{-1}]$ is a class of "roofs", i.e., of diagrams (s, f) in $\mathcal{C}$ of the form

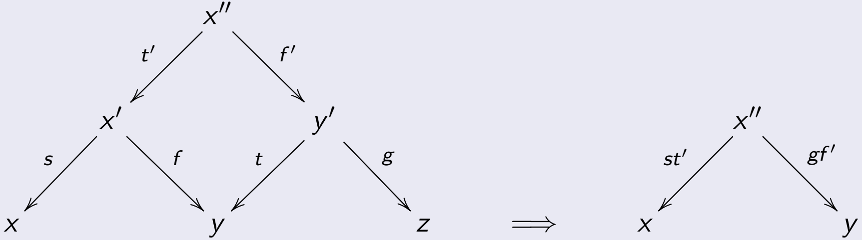
$$\begin{array}{ccc} & x' & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} \\ x & & y\,, \end{array}$$

where $s \in \Sigma$ and $f \in \mathcal{C}$.

Two roofs $(s, f) \sim (t, g)$ *are equivalent, if and only if, there is a third roof* (r, h) *with*



The identities are $(1_x, 1_x)$ *and the composition of* (s, f) *and* (t, g) *is*



Steps to find $\pi_1(\mathscr{S}^G)$

1. We consider the subset of morphisms $\tilde{\mathscr{S}}_0^G$ which are the disjoint union of identity G-cobordisms over cylinders and G-cobordisms over closed surfaces.
2. $\tilde{\mathscr{S}}_0^G$ is a localizing set. Therefore, the category of fractions $\mathscr{S}^G[(\tilde{\mathscr{S}}_0^G)^{-1}]$ can be described by roofs

$$(\mathrm{id}_{\hat{x}} \sqcup \gamma, \Sigma) \quad = \quad \begin{array}{ccccc} & & \hat{x} & & \\ & \swarrow^{\mathrm{id}_{\hat{x}} \sqcup \gamma} & & \searrow^{\Sigma} & \\ \hat{x} & & & & \hat{y}\,, \end{array}$$

where $\Sigma : \hat{x} \to \hat{y}$ is a morphism in $\mathscr{S}^G$ and γ is a G-cobordism over closed surfaces.

3. We restrict to objects of $\mathscr{S}^G$ which are in the connected component of the empty G-bundle.
4. For a morphism $\Sigma : \hat{x} \to \hat{y}$ we consider $\delta_{\hat{y}}$ a connected morphism from $\hat{y}$ to 0. Set $\Sigma' = \Sigma \sqcup_{\hat{y}} \delta_{\hat{y}}$. The proposal for the inverse of $(\mathrm{id}_{\hat{x}}, \Sigma)$ is

$$\Theta := \left(\mathrm{id}_{\hat{y}} \sqcup (\Sigma' \circ \overline{\Sigma'}), \overline{\Sigma'} \circ \delta_{\hat{y}}\right)$$

where $\overline{\Sigma'}$ is Σ' with the reverse orientation.
5. The composition of $(\mathrm{id}_{\hat{x}}, \Sigma)$ followed by Θ is

$$\left(\mathrm{id}_{\hat{x}} \sqcup (\Sigma' \circ \overline{\Sigma'}), \overline{\Sigma'} \circ \Sigma'\right)$$

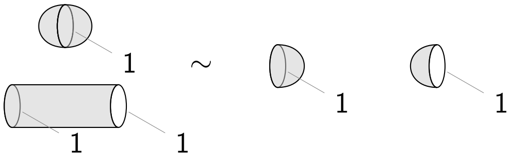
6. Consider the equivalence relation "generated" by the identification

$$\mathrm{id}_{\hat{x}} \sqcup (\Sigma' \circ \overline{\Sigma'}) \sim \overline{\Sigma'} \circ \Sigma'$$

7. Denote by $\mathscr{S}^G_\sim$ the quotient category given by the equivalence relation (recall we are in the connected component of the empty G-bundle).

8. There is an isomorphism $\mathscr{S}^G[\mathscr{S}^{G^{-1}}] \cong \mathscr{S}^G_\sim[(\tilde{\mathscr{S}}^G_0)^{-1}]$.

9. The relation $\mathrm{id}_{\hat{X}} \sqcup (\Sigma \circ \overline{\Sigma}) \sim \overline{\Sigma} \circ \Sigma$ is implied from the relation $\mathrm{id}_1 \sqcup (D \circ \overline{D}) \sim \overline{D} \circ D$ for D the disc $D : 1 \to 0$.



The splitting $\pi_1(\mathscr{S}^G) = \mathbb{Z} \oplus \Omega_2^{SO}(G)$

- The fundamental group $\pi_1(\mathscr{S}^G)$ can be identified with the automorphism group $\mathrm{Hom}_{\mathscr{S}^G[\mathscr{S}^{G^{-1}}]}(0,0)$. This monoid consists of roofs (Γ, Σ) and the composition is

$$(\Gamma_2, \Sigma_2) \circ (\Gamma_2, \Sigma_2) = (\Gamma_1 + \Gamma_2, \Sigma_1 + \Sigma_2)$$

- We define the homomorphism $\pi_1(\mathscr{S}^G) \to \Omega_2^{SO}(G)$ by $(\Gamma, \Sigma) \longmapsto [\Sigma] - [\Gamma]$, which is well defined.
- The splitting $\Omega_2^{SO}(G) \to \pi_1(\mathscr{S}^G)$ starts with a G-cobordism and we separate along every simple closed curve with trivial monodromy and we cap with two discs, where every time we separate we add the negative of the G-cobordism over the sphere.

Theorem

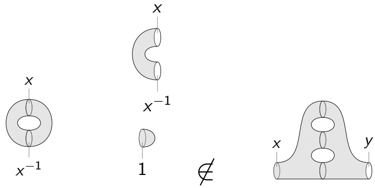
We have the splitting short exact sequence

$$0 \to \mathbb{Z} \to \pi_1(\mathscr{S}^G) \to \Omega_2^{SO}(G) \to 0$$

The classifying space $B\mathscr{S}^G$

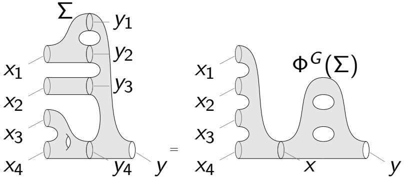
- Similarly, we define the subcategories

$$\mathscr{S}_0^G \qquad \mathscr{S}_{>0}^G \qquad \mathscr{S}_1^G$$



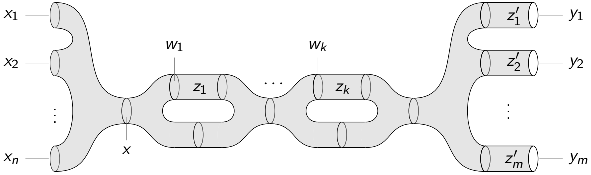
- We have the adjoint functors $\Phi^G : \mathscr{S}_{>0}^G \to \mathscr{S}_1^G \;\dashv\; i : \mathscr{S}_1^G \hookrightarrow \mathscr{S}_{>0}^G$

$$\begin{array}{ccc} (x_1, \cdots, x_n) & \xrightarrow{P_{(x_1,\cdots,x_n)}} & x = \prod_{i=1}^n x_i \\ \Sigma \downarrow & & \downarrow \Phi^G(\Sigma) \\ (y_1, \cdots, y_m) & \xrightarrow{P_{(y_1,\cdots,y_m)}} & y = \prod_{j=1}^m y_j \end{array}$$



$$\Phi^G \circ i = \mathrm{id}_{\mathscr{S}_1^G} \text{ and } i \circ \Phi^G \simeq \mathrm{id}_{\mathscr{S}_{>0}^G}$$

- **Assumption!** Every G-cobordism with connected base space and non-empty incoming boundary $\hat{x} = (x_1, \cdots, x_n)$, with $x_i \neq 0$ for $1 \leq i \leq n$, factorises through the precomposition of the G-cobordism $P_{\hat{x}}$ (the pair of pants with multiple legs).



Theorem (Tillmann)

$B\mathscr{S}_0^G \simeq T^\infty$, $B\mathscr{S}_1^G, B\mathscr{S}_{>0}^G, B\mathscr{S}_b^G \simeq B\mathscr{S}_1^G$ and

$$B\mathscr{S}^G \simeq \frac{G}{[G,G]} \times X^G \times S^1$$

where X^G is an infinite loop space with $\pi_1(X^G) = \Omega_2^{SO}(G)$.

Applications

- Bökstedt-Svane: $\pi_1(\mathcal{C}_2^G)$ is generated by diffeomorphism classes $[W]$ closed up to the *"Chimera relations"*

$$
\begin{aligned}
&(\mathrm{id}_0, W_3 \circ \overline{W_1} \sqcup W_4 \circ \overline{W_2}) = \\
&= \big(W_1 \circ \overline{W_1} \sqcup W_2 \circ \overline{W_2}, (W_3 \circ \overline{W_2} \circ W_2 \circ \overline{W_1}) \sqcup (W_4 \circ \overline{W_1} \circ W_1 \circ \overline{W_2})\big) \\
&= \big(W_1 \circ \overline{W_1} \sqcup W_2 \circ \overline{W_2}, (W_3 \circ \overline{W_2} \sqcup W_2 \circ \overline{W_1}) \sqcup (W_4 \circ \overline{W_1} \sqcup W_1 \circ \overline{W_2})\big) \\
&= \big(W_1 \circ \overline{W_1} \sqcup W_2 \circ \overline{W_2}, (W_3 \circ \overline{W_2} \sqcup W_4 \circ \overline{W_1}) \sqcup (W_2 \circ \overline{W_1} \circ W_1 \circ \overline{W_2})\big) \\
&= \big(\mathrm{id}_0, W_3 \circ \overline{W_2} \sqcup W_4 \circ \overline{W_1}\big)
\end{aligned}
$$

We ignore if $\pi_2(\mathcal{C}_2^G) \to \pi_2(\mathscr{S}^G)$ is an epimorphism.

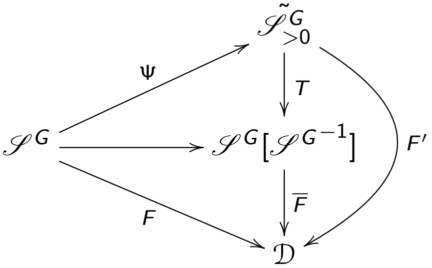
- Schneiden-Kleben bordism:

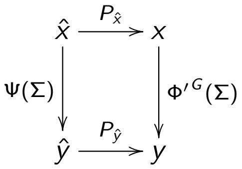
$$
\begin{CD}
@. 0 @>>> (\text{Toral}) @>>> (\text{Toral}) @>>> 0 \\
@. @VVV @VVV @VVV @. \\
0 @>>> \mathbb{Z} @>>> \pi_1(\mathscr{S}^G) @>>> \Omega_2^{SO}(G) @>>> 0 \\
@. @VVV @VVV @VVV @. \\
0 @>>> \mathbb{Z} @>>> \pi_1(\mathscr{S}^G_{SK}) @>>> B_2(G) @>>> 0 \\
@. @VVV @VVV @VVV @. \\
@. 0 @. 0 @. 0 @.
\end{CD}
$$

$$\pi_1(\mathscr{S}^G_{SK}) = SK_2(G) = \widetilde{SK}_2(G) \oplus SK_2 = B_0(G) \oplus \mathbb{Z}$$

G-TQFT

- For **a** functor $F : \mathscr{S}^G \to \mathfrak{D}$ with $\mathfrak{D}$ a groupoid





$$F(\Sigma) = F'(P_{\hat{y}})^{-1} \circ F'(\Phi'^G(\Sigma)) \circ F'(P_{\hat{x}})$$

For $\mathfrak{D}$ and abelian group and $\Sigma : m \to n$, we obtain

$$F(\Sigma) = c_n - c_m + a_0 \Phi(\Sigma),$$

where $c_n = F(P_n)$ and $a_0 = F(S^2)$.

- $F : \mathscr{S}^G \to \mathrm{Vect}_{\mathbb{C}}$ an **invertible** symmetric monoidal functor. We obtain the discrete torsion determined by Turaev

$$\mathbb{C}_b(G) := \bigoplus_{x \in G} \mathbb{C} \times \{x\}$$

where for a basis e_x, we have a 2-cocycle $b \in H^2(G, \mathbb{C}^*)$ with $e_x \cdot e_y = b(x, y) \cdot e_{xy}$. We have a diagram

$$\begin{array}{ccc} \mathscr{S}^G & \longrightarrow & \mathscr{S}^G[\mathscr{S}^{G^{-1}}] \\ & \searrow^{F} & \downarrow \\ & & \mathrm{Vect}_{\mathbb{C}} \end{array}$$

Thus F is completely determined by the restriction to the automorphism group of the empty bundle in $\mathscr{S}^G[\mathscr{S}^{G^{-1}}]$, plus the images of a generating set of the monoid $\mathcal{M}_G$ of "handles". This induces a representation of the fundamental group $\mathbb{Z} \oplus \Omega_2^{SO}(G) \to \mathbb{C}^*$.

- For **any** symmetric monoidal functor $F : \mathscr{S}^G \longrightarrow \mathrm{Vect}_{\mathbb{C}}$, we consider again the subset of morphisms $\tilde{\mathscr{S}}^G_{>0}$ of those endomorphisms in $\mathscr{S}^G$ which are the disjoint union of identity G-cobordisms over cylinder and G-cobordisms over closed surfaces. Any G-cobordisms over a closed surfaces can be written as a double construction $M \sqcup_X \underline{M}$. Non-degeneracy implies that we assign a non-zero complex number to each closed G-cobordisms. Thus we have the diagram

$$\begin{array}{ccc} \mathscr{S}^G & \longrightarrow & \mathscr{S}^G[\tilde{\mathscr{S}}^{G}_{0}{}^{-1}] \\ & {}_{F}\searrow & \downarrow \\ & & \mathrm{Vect}_{\mathbb{C}} \end{array}$$

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