Symmetrization for Fractional Elliptic Problems: A Direct Approach

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a Bridge between PDEs, Analysis and Geometry

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Symmetrization for local problems

Let us consider the following homogeneous Dirichlet problem in an open bounded set $\Omega \subset \mathbb{R}^N$, $N \geq 2$,

$$\begin{cases} -\left(a_{ij}\ u_{x_i}\right)_{x_j} = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the measurable coefficients $a_{ij} = a_{ij}(x)$ satisfy the ellipticity condition

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \qquad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega,$$

and the source term $f = f(x)$ is assumed to belong to $L^p(\Omega)$ for suitable $p \geq 1$.

Symmetrization for local problems

A nowadays classical result states that if $u \in H_0^1(\Omega)$ is the weak solution to (1) and $v \in H_0^1(\Omega^*)$ is the weak solution to the "symmetrized problem"

$$\begin{cases} -\Delta v = f^* & \text{in } \Omega^*, \\ v = 0 & \text{on } \partial\Omega^*, \end{cases}$$

then

$$u^*(x) \le v(x), \qquad x \in \Omega^*. \tag{2}$$

Here Ω^* is the ball centered at the origin such that $|\Omega^*| = |\Omega|$ and u^* denotes the Schwarz symmetrization of u:

$$u^*(x) = \sup\{t \ge 0 : |\{x : |u(x)| > t\}| > \omega_N |x|^N\} \ (= u^*(|x|)),$$

where ω_N is the measure of the unit ball in $\mathbb{R}^N$.

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where ω_N is the measure of the unit ball in $\mathbb{R}^N$.
An immediate consequence of inequality (2) is, for example, that any norm of u increases under Schwarz symmetrization.

[Weinberger, 1962], [Maz'ya, 1971], [Talenti, 1976]

Symmetrization for local problems

The approach used in most of the papers concerning symmetrization techniques is based on the fact that the use of a suitable test function allows to obtain, for a.e. $t \in (0, \sup|u|)$, the inequality

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}|Du|^2\mathrm{d}x \le \int_{u^*>t} f^*(x)\,\mathrm{d}x. \tag{3}$$

Schwarz inequality, Fleming-Rishel formula and isoperimetric inequality are then used in order to obtain a first order differential inequality involving u^* and its radial derivative. Finally, a comparison principle gives

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A slightly different approach has been used in [Lions, 1981], where the author observes that in inequality (3) one can use the so-called Pólya-Szegö principle which states that, if $u \in H_0^1(\Omega)$, then

$$\int_\Omega |Du|^2 \mathrm{d}x \geq \int_\Omega |Du^*|^2 \mathrm{d}x. \tag{4}$$

Symmetrization for local problems

The differential quotient used to compute the derivative

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{|u|>t}|Du|^2\mathrm{d}x$$

is given by ($h>0$)

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that is, the Dirichlet integral of a truncation of u, which is a Sobolev function.

So Pólya-Szegö principle applies to give

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{u^*>t}|Du^*|^2\mathrm{d}x \leq \int_{u^*>t} f^*(x)\,\mathrm{d}x.$$

At this point the integral on the left hand side concerns a radially symmetric function and the quoted first order differential inequality involving u^* follows immediately, without the use of isoperimetric inequality.

Symmetrization for local problems

Actually, for every r such that $u^*(r) = t$, co-area formula gives

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{u^*>t}|Du^*|^2\mathrm{d}x = \int_{u^*=t}|Du^*|\mathrm{d}\sigma = \mathrm{Per}(B_r)\left(-\frac{\mathrm{d}}{\mathrm{d}r}u^*(r)\right)$$

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and finally

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The solution v to the symmetrized problem satisfies

$$-\frac{\mathrm{d}}{\mathrm{d}r}v^*(r) = \frac{1}{N\omega_N r^{N-1}}\int_{B_r} f^*(x)\,\mathrm{d}x$$

and the comparison follows immediately.

Symmetrization for local problems

The literature about the possible extensions of the comparison result is wide

- elliptic equations with lower order terms
- p-Laplacian type equations
- porous medium equation
- parabolic equations
- anisotropic equations
- ...

Symmetrization for nonlocal problems

Let us consider the following Dirichlet fractional elliptic problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a smooth bounded open set, the source term $f = f(x)$ is assumed to belong to $L^p(\Omega)$ for suitable $p \geq 1$ and $s \in (0,1)$.

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In those papers a symmetrization result in terms of mass concentration (*i.e.*, an integral comparison, as in the parabolic case) is obtained in a somewhat indirect way.

Symmetrization for nonlocal problems

Indeed, it has been used in an essential way the fact that the fractional problem can be linked to a suitable, local extension problem, whose solution $\psi(x, y)$, an extension of u, is defined on the infinite cylinder $\mathcal{C}_\Omega = \Omega \times (0, +\infty)$.

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Being u the trace of ψ over $\Omega \times \{0\}$, the comparison result for ψ immediately implies an estimate for u

Symmetrization for nonlocal problems

Theorem ([F. -Volzone, 2021])

Let $s \in (0,1)$ *and let* $f \in L^p(\Omega)$, *with* $p \geq 2N/(N+2s)$ *when* $N \geq 2$ *and any* $p > 1$ *for* $N = 1$. *If* u *and* v *are the solutions to the following problems*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases} \qquad \begin{cases} (-\Delta)^s v = f^* & \text{in } \Omega^* \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega^* \end{cases}$$

we have

$$u(x) \prec v(x)$$

and

$$[u]_{H^s} \leq [v]_{H^s}$$

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$$u(x) \prec v(x) \qquad (\textit{comparison of mass concentrations})$$

means that for all $r > 0$ it holds

$$\int_{B_r(0)} u^*(x)\,\mathrm{d}x \leq \int_{B_r(0)} v^*(x)\,\mathrm{d}x$$

Symmetrization for nonlocal problems

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As we will see our method is based on a suitable Pólya-Szegö principle and, because of the fact that such a principle holds true in more general situations, the extension to various classes of nonlocal PDEs seems to be possible.

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Possible examples in the elliptic framework are nonlocal semilinear equations, equations involving elliptic integro-differential operators with general kernels, fractional p-Laplacian operator.

Optimality of the result

One could ask if the comparison in terms of mass concentration could be improved to give a pointwise estimate. In order to understand if a result similar to the one proved by Talenti we have considered, in the case $N = 1$, $s \in (0,1)$, $\Omega = (-1,1)$, the problem

$$\begin{cases} (-\Delta)^s u = |x| & \text{in } \Omega \\ u = 0 & \text{on } \mathbb{R} \setminus \Omega \end{cases}$$

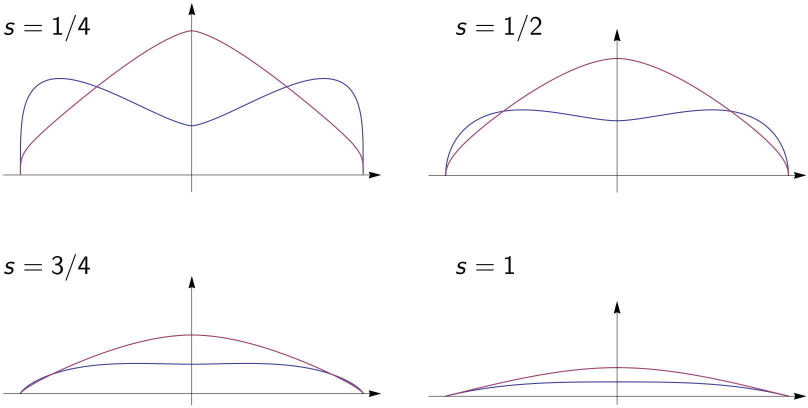
Clearly we have

$$f(x) = |x|, \qquad f^*(x) = 1 - |x| = 1 - f(x)$$

We denote by u_s the solution to the given problem and by v_s the solution to the corresponding symmetrized problem.

Optimality of the result

u_s (blue line) $\qquad v_s$ (purple line)



Sketch of the proof

Step 1: Deduce an inequality for u^* via Riesz rearrangement inequality

For simplicity we will consider f nonnegative and regular.
In the weak formulation of the problem

$$\frac{\gamma(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y=\int_\Omega f(x)\,\varphi(x)\,\mathrm{d}x$$

for $0 \le t < u_{\max}$ and $h > 0$, we choose the following test function

$$\varphi(x) = \mathcal{G}_{t,h}(u(x))$$

where $\mathcal{G}_{t,h}(\theta)$ is the classical truncation

$$\mathcal{G}_{t,h}(\theta)=\begin{cases} h & \text{if } \theta > t+h \\ \theta - t & \text{if } t < \theta \le t+h \\ 0 & \text{if } \theta \le t. \end{cases}$$

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Theorem (Riesz rearrangement inequality [Almgren - Lieb, 1989])

Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous function such that* $F(0,0) = 0$ *and*

$$F(u_2, v_2) + F(u_1, v_1) \geq F(u_2, v_1) + F(u_1, v_2)$$

whenever $u_2 \geq u_1 > 0$ *and* $v_2 \geq v_1 > 0$*. Assume that* f, g *are nonnegative measurable functions on* $\mathbb{R}^N$*, then we have the inequalities*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(f(x), g(y)) W(ax + by)\, dx\, dy \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} F(f^*(x), g^*(y)) W^*(ax + by)\, dx\, dy$$

and

$$\int_{\mathbb{R}^N} F(f(x), g(x))\, dx \leq \int_{\mathbb{R}^N} F(f^*(x), g^*(x))\, dx,$$

for any nonnegative function $W \in L^1(\mathbb{R}^N)$ *and any choice of nonzero numbers* a *and* b*.*

Sketch of the proof

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Our aim is to prove

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))\,(\mathcal{G}_{t,h}(u(x))-\mathcal{G}_{t,h}(u(y)))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u^*(x)-u^*(y))\,(\mathcal{G}_{t,h}(u^*(x))-\mathcal{G}_{t,h}(u^*(y)))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

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$$\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u^*(x)-u^*(y))\,(\mathcal{G}_{t,h}(u^*(x))-\mathcal{G}_{t,h}(u^*(y)))}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y.$$

We use the representation

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))\,(\mathcal{G}_{t,h}(u(x))-\mathcal{G}_{t,h}(u(y)))}{|x-y|^{N+2s}}\mathrm{d}x\,\mathrm{d}y =$$

$$= \frac{1}{\Gamma(\frac{N+2s}{2})}\int_0^\infty I_\alpha[u,t,h]\,\alpha^{(N+2s)/2-1}\mathrm{d}\alpha,$$

where

$$I_\alpha[u,t,h] = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))\,(\mathcal{G}_{t,h}(u(x))-\mathcal{G}_{t,h}(u(y)))\exp[-|x-y|^2\alpha]\mathrm{d}x\,\mathrm{d}y.$$

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Step 1: Deduce an inequality for u^* via Riesz rearrangement inequality

By virtue of this last representation, our claim is proved when we succeed to show that

$$I_\alpha[u,t,h] \geq I_\alpha[u^*,t,h],$$

for all $\alpha > 0$. To this aim, we use Riesz's general rearrangement inequality with the choice $W_\alpha(x) = \exp[-|x|^2\alpha]$, $a = 1$, $b = -1$ and

$$F(u,v) = u^2 + v^2 - (u-v)(\mathcal{G}_{t,h}(u) - \mathcal{G}_{t,h}(v))$$

for all u, $v > 0$, with W_α and $F(u,v)$ which satisfy the required assumptions.

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for all u, $v > 0$, with W_α and $F(u,v)$ which satisfy the required assumptions. So we obtain

$$\frac{\gamma(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u^*(x)-u^*(y))\,(\mathcal{G}_{t,h}(u^*(x))-\mathcal{G}_{t,h}(u^*(y)))}{|x-y|^{N+2s}}\,dx\,dy \leq$$

$$\leq \int_\Omega f(x)\,\mathcal{G}_{t,h}(u(x))\,dx.$$

Sketch of the proof

Step 2: Pass to the limit as $h \to 0$

This step is quite technical and, writing $u^*(x) = u^*(|x|)$, we get, for $r > 0$,

$$\gamma(N,s)\int_0^r \left(\int_r^{+\infty} (u^*(\tau) - u^*(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\mathrm{d}\rho\right)\tau^{N-1}\mathrm{d}\tau \le$$

$$\le \int_0^r f^*(\rho)\rho^{N-1}\mathrm{d}\rho,$$

where

$$\Theta_{N,s}(r,\rho) = \frac{1}{N\omega_N}\int_{|x'|=1}\left(\int_{|y'|=1}\frac{1}{|r\,x' - \rho\,y'|^{N+2s}}\mathrm{d}H^{N-1}(y')\right)\mathrm{d}H^{N-1}(x')$$

that is,

$$\Theta_{N,s}(r,\rho) = \begin{cases} \dfrac{\alpha_N}{\rho^{N+2s}}\ {}_2F_1\left(\dfrac{N+2s}{2}, s+1; \dfrac{N}{2}; \dfrac{r^2}{\rho^2}\right) & \text{if } 0 \le r < \rho < +\infty \\[2ex] \dfrac{\alpha_N}{r^{N+2s}}\ {}_2F_1\left(\dfrac{N+2s}{2}, s+1; \dfrac{N}{2}; \dfrac{\rho^2}{r^2}\right) & \text{if } 0 \le \rho < r < +\infty \end{cases}$$

Sketch of the proof

Step 3: Rewriting the above inequality in terms of the spherical mean function

Let us define the following spherical mean function

$$U(x) = U(|x|) = \frac{1}{|x|^N} \int_0^{|x|} u^*(\rho)\rho^{N-1} \mathrm{d}\rho.$$

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It turns out that

$$\gamma(N,s) \int_0^r \left(\int_r^{+\infty} \big(u^*(\tau) - u^*(\rho)\big) \Theta_{N,s}(\tau,\rho)\rho^{N-1} \mathrm{d}\rho \right) \tau^{N-1} \mathrm{d}\tau = r^N (-\Delta)^s_{\mathbb{R}^{N+2}} U(r)$$

A formula for the fractional Laplacian computed on radial function contained in [Ferrari - Verbitsky, 2012] has been used.

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A formula for the fractional Laplacian computed on radial function contained in [Ferrari - Verbitsky, 2012] has been used.
Then

$$(-\Delta)^s_{\mathbb{R}^{N+2}} U(r) \leq \frac{1}{r^N} \int_0^r f^*(\rho)\rho^{N-1} \mathrm{d}\rho$$

Sketch of the proof

Step 4: Comparison principle and end of the proof

For the solution v to the symmetrized problem all the above inequalities hold true as equalities, so

$$(-\Delta)^s_{\mathbb{R}^{N+2}} V(r) = \frac{1}{r^N}\int_0^r f^*(\rho)\rho^{N-1}\mathrm{d}\rho$$

where $V(r)$ is the spherical mean of v,

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A classical comparison result gives

$$U(r) \leq V(r)$$

that is,

$$u \prec v.$$

A remark

A way to recover the pointwise comparison is to observe that, letting $s \to 1$, we obtain a comparison between local Laplacians in the form

$$(-\Delta)_{\mathbb{R}^{N+2}} U(r) \leq (-\Delta)_{\mathbb{R}^{N+2}} V(r)$$

and a straightforward computation shows

$$(-\Delta)_{\mathbb{R}^{N+2}} U(r) = -\frac{{u^*}'(r)}{r}, \quad (-\Delta)_{\mathbb{R}^{N+2}} V(r) = -\frac{v'(r)}{r}$$

from which the pointwise comparison follows.