Stability in Gagliardo-Nirenberg-Sobolev inequalities 1/3: A variational point of view

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September 8th, 2021

BIRS-CMO workshop
New Trends in Nonlinear Diffusion: a Bridge between PDEs, Analysis and Geometry
Oaxaca (September 5-10, 2021)

Joint work on *Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method* arXiv:2007.03674 (Apr. 29, 2021) with

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Outline and reference

Chapter 1: Gagliardo-Nirenberg-Sobolev inequalities by variational methods

▷ Gagliardo-Nirenberg-Sobolev inequalities

▷ Relative entropy and relative Fisher information

▷ Optimality in GNS inequalities

▷ A stability result for GNS inequalities

with Matteo Bonforte, Jean Dolbeault, Nikita Simonov

M. Bonforte, J. Dolbeault, B. Nazaret, and N. Simonov. *Stability in Gagliardo-Nirenberg-Sobolev inequalities.*
Preprint https://hal.archives-ouvertes.fr/hal-02887010

- We consider the inequalities

$$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta} \geq \mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p} \qquad \text{(GNS)}$$

$$\theta = \frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}\,,\quad p\in(1,+\infty) \text{ if } d=1 \text{ or } 2\,,\quad p\in(1,p^*] \text{ if } d\geq 3\,,\quad p^*=\frac{d}{d-2}$$

▷ Function space $\mathscr{H}_p(\mathbb{R}^d)$: completion of $\mathscr{C}_c^\infty(\mathbb{R}^d)$ with respect to the norm

$$f\mapsto (1-\theta)\,\|f\|_{p+1}+\theta\,\|\nabla f\|_2\,.$$

- [del Pino, Dolbeault (2002)]: Equality case in (GNS) is achieved if and only if

$$f\in\mathfrak{M} := \left\{g_{\lambda,\mu,y} \,:\, (\lambda,\mu,y)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\right\}$$

▷ *Aubin-Talenti* functions:

$$g_{\lambda,\mu,y}(x) := \mu\,\mathbf{g}\left((x-y)/\lambda\right),\quad \mathbf{g}(x)=\left(1+|x|^2\right)^{-\frac{1}{p-1}}$$

Related inequalities

$$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta} \geq \mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p} \qquad \text{(GNS)}$$

- *Sobolev's inequality* : $d \geq 3$, $p = p^* = d/(d-2)$

$$\mathsf{S}_d\,\|\nabla f\|_2 \geq \|f\|_{2p^*}$$

- *Euclidean Onofri inequality*

$$\int_{\mathbb{R}^2} e^{h-\overline{h}}\,\frac{dx}{\pi\,(1+|x|^2)^2} \leq e^{\frac{1}{16\pi}\int_{\mathbb{R}^2}|\nabla h|^2\,dx}$$

$d = 2$, $p \to +\infty$ with $f_p(x) := \mathrm{g}(x)\left(1 + \frac{1}{2p}\,(h(x)-\overline{h})\right)$, $\overline{h} = \int_{\mathbb{R}^2} h(x)\,\frac{dx}{\pi\,(1+|x|^2)^2}$

- *Euclidean logarithmic Sobolev inequality in scale invariant form*

$$\frac{d}{2}\log\left(\frac{2}{\pi\,d\,e}\int_{\mathbb{R}^d}|\nabla f|^2\,dx\right) \geq \int_{\mathbb{R}^d}|f|^2\log|f|^2\,dx$$

or $\int_{\mathbb{R}^d}|\nabla f|^2\,dx \geq \frac{1}{2}\int_{\mathbb{R}^d}|f|^2\log\left(\frac{|f|^2}{\|f\|_2^2}\right)dx + \frac{d}{4}\log\left(2\pi\,e^2\right)\int_{\mathbb{R}^d}|f|^2\,dx$

Deficit functional, scale invariance

Deficit functional

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GNS}\,\|f\|_{2p}^{2p\gamma}$$

Lemma

(GNS) *is equivalent to* $\delta[f] \ge 0$ *if and only if*

$$\mathscr{K}_{\rm GNS} = C(p,d)\,\mathscr{C}_{\rm GNS}^{2p\gamma}$$

where $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$ *and* $C(p,d)$ *is an explicit positive constant*

Take $f_\lambda(x) = \lambda^{\frac{d}{2p}}\,f(\lambda\,x)$ and optimize on $\lambda > 0$

$$\delta[f] \ge \delta[f] - \inf_{\lambda>0} \delta[f_\lambda] =: \delta_\star[f] \ge 0$$

A simplification: $\delta[f] = \delta[|f|]$ so we shall assume that $f \ge 0$ a.e.

Existence of an optimal function

$$I_M = \inf\left\{(p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} : f \in \mathscr{H}_p(\mathbb{R}^d),\quad \|f\|_{2p}^{2p} = M\right\}$$

$$I_1 = \mathscr{K}_{\rm GNS} \text{ and } I_M = I_1\,M^\gamma \text{ for any } M>0$$

Lemma

If $d \geq 1$ and p is an admissible exponent with $p < d/(d-2)$, then

$$I_{M_1+M_2} < I_{M_1} + I_{M_2} \quad \forall\, M_1, M_2 > 0$$

Lemma

Let $d \geq 1$ and p be an admissible exponent with $p < d/(d-2)$ if $d \geq 3$. If $(f_n)_n$ is minimizing and if $\limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^d} \int_{B(y)} |f_n|^{p+1}\,dx = 0$, then

$$\lim_{n\to\infty} \|f_n\|_{2p} = 0$$

Existence of a minimizer, further properties

Proposition

Assume that $d \geq 1$ is an integer and let p be an admissible exponent with $p < d/(d-2)$ if $d \geq 3$. Then there is an optimal function for (GNS)

- *Pólya-Szegö principle*: there is a radial minimizer solving

$$-2\,(p-1)^2\,\Delta f + 4\,\big(d - p\,(d-2)\big)\,f^p - C\,f^{2\,p-1} = 0$$

If $f = \mathbf{g}$, then $C = 8\,p$

- *A rigidity result*: if f is a (smooth) minimizer and $\mathrm{P} = -\frac{p+1}{p-1}\,f^{1-p}$, then

$$\big(d - p\,(d-2)\big)\int_{\mathbb{R}^d} f^{p+1}\left|\Delta\mathrm{P} + (p+1)^2\,\frac{\int_{\mathbb{R}^d}|\nabla f|^2\,\mathrm{d}x}{\int_{\mathbb{R}^d} f^{p+1}\,\mathrm{d}x}\right|^2\mathrm{d}x$$
$$+\,2\,d\,p\int_{\mathbb{R}^d} f^{p+1}\left\|\mathrm{D}^2\mathrm{P} - \tfrac{1}{d}\,\Delta\mathrm{P}\,\mathrm{Id}\right\|^2\mathrm{d}x = 0$$

▷ $\mathbf{g}$ is a minimizer and $\delta[\mathbf{g}] = 0$

Relative entropy and Fisher information

- *Free energy* or *relative entropy functional*

$$\mathscr{E}[f|g] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(f^{p+1} - g^{p+1} - \tfrac{1+p}{2p}\, g^{1-p} \left(f^{2p} - g^{2p} \right) \right) dx \geq 0$$

Lemma (Csiszár-Kullback inequality)

Let $d \geq 1$ and $p > 1$. There exists a constant $C_p > 0$ such that

$$\left\| f^{2p} - g^{2p} \right\|^2_{L^1(\mathbb{R}^d)} \leq C_p \mathscr{E}[f|g] \quad \textit{if} \quad \|f\|_{2p} = \|g\|_{2p}$$

- *Relative Fisher information*

$$\mathscr{J}[f|g] := \frac{p+1}{p-1} \int_{\mathbb{R}^d} \left| (p-1) \nabla f + f^p \nabla g^{1-p} \right|^2 dx$$

Best matching profile

- Nonlinear extension of the *Heisenberg uncertainty principle*

$$\left(\frac{d}{p+1}\int_{\mathbb{R}^d} f^{p+1}\,dx\right)^2 \le \int_{\mathbb{R}^d} |\nabla f|^2\,dx \int_{\mathbb{R}^d} |x|^2 f^{2p}\,dx$$

▷ Take $g = \mathbf{g}$ in $\mathcal{J}[f|g]$ and expand the square.

- If

$$\int_{\mathbb{R}^d} f^{2p}\,(1, x, |x|^2)\,dx = \int_{\mathbb{R}^d} g^{2p}\,(1, x, |x|^2)\,dx\,, \quad g \in \mathfrak{M} \tag{1}$$

$$\text{then} \quad \mathcal{E}[f|g] = \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(f^{p+1} - g^{p+1}\right)dx$$

▷ A smaller space: $\mathcal{W}_p(\mathbb{R}^d) := \{f \in \mathcal{H}_p(\mathbb{R}^d) : |x|\,|f|^p \in \mathrm{L}^2(\mathbb{R}^d)\}$

Lemma

For any $f \in \mathcal{W}_p(\mathbb{R}^d)$, we have

$$\mathcal{E}[f|\mathrm{g}_f] = \inf_{g \in \mathfrak{M}} \mathcal{E}[f|g]\,,$$

where g_f is the unique function in $\mathfrak{M}$ satisfying (1)

A first (weak) stability result

Lemma (A weak stability result)

If $g_f = \mathbf{g}$, *then*

$$\delta[f] \geq \delta_\star[f] \approx \mathscr{E}[f|\mathbf{g}]^2$$

▷ Up to the invariances, $\mathbf{g}$ is the unique minimizer for $f \mapsto \delta[f]$, hence for (GNS)

Lemma (Entropy - entropy production inequality)

If $\|f\|_{2p} = \|g\|_{2p}$ *with* $\delta[g] = 0$,

$$\frac{p+1}{p-1}\,\delta[f] = \mathscr{J}[f|g] - 4\,\mathscr{E}[f|g] \geq 0$$

▷ From now on, we will assume that $g_f = \mathbf{g}$, *i.e.*

$$\int_{\mathbb{R}^d} f^{2p}\,(1, x, |x|^2)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathbf{g}^{2p}\,(1, x, |x|^2)\,\mathrm{d}x\,,$$

- [Bianchi, Egnell (1991)] There is a positive constant α such that

$$\mathsf{S}_d \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \alpha \inf_{\varphi \in \mathscr{M}} \|\nabla f - \nabla \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- Various extensions to:

▷ L^q norm of the gradient by [Chianchi, Fusco, Maggi, Pratelli (2009)], [Figalli, Neumayer (2018)], [Neumayer (2020)], [Figalli, Zhang (2020)]

▷ (GNS) inequalities by [Carlen, Figalli (2013)], [Seuffert (2017)], [Nguyen (2019)]

Theorem

There exists a constant $C > 0$ such that

$$\delta[f] \geq C\,\mathscr{E}[f|\mathbf{g}]$$

for any $f \in \mathscr{W}_p(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} f^{2p}\,(1, x, |x|^2)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathbf{g}^{2p}\,(1, x, |x|^2)\,\mathrm{d}x\,,$$

Proof using spectral gap

- The spectral gap inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, \mathbf{g}^{2p} \, dx \geq \frac{4p}{p-1} \int_{\mathbb{R}^d} |u|^2 \, \mathbf{g}^{3p-1} \, dx$$

valid for any function u such that $\int_{\mathbb{R}^d} u \, \mathbf{g}^{3p-1} \, dx = 0$, can be improved with a constant $\Lambda_\star > 4p/(p-1)$ under the constraint that

$$\int_{\mathbb{R}^d} \left(1, x, |x|^2\right) u(x) \, \mathbf{g}(x)^{3p-1} \, dx = 0$$

- A Taylor expansion with $f = \mathbf{g} + \eta h$ gives

$$\lim_{\eta \to 0} \frac{\delta[f_\eta]}{\mathscr{E}[f_\eta|\mathrm{g}]} \geq \frac{(p-1)^2}{p(p+1)} \left[\Lambda_\star - \frac{4p}{p-1}\right]$$

Thank you for your attention !