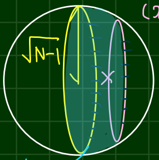
Spectral convergence of $\{\mathbb{S}^N(\sqrt{N-1})\}$ to $\Gamma^k = (\mathbb{R}^k, \gamma^k)$

Ric $\equiv 1$ & dim $= N$ Ric $\equiv 1$ & dim $= \infty$

$\mathbb{S}^N(\sqrt{N-1}) \subset \mathbb{R}^{N+1} = \mathbb{R}^k \times \mathbb{R}^{N-k+1}$

$= (2\pi)^{-\frac{k}{2}} \exp\left(-\frac{|x|^2}{2}\right) dx$



$\xrightarrow{N \to \infty} \mathbb{R}^\infty$?

$A \subset \mathbb{R}^k$

$$v_N(A) := \frac{|(\mathrm{proj}^N_k)^{-1}(A)|}{|\mathbb{S}^N(\sqrt{N-1})|}$$

$\downarrow N \to \infty$

$\gamma^k(A)$ (Boltzmann-Maxwell distribution law)

$(\mathrm{proj}^N_k)^{-1}(A)$

$\downarrow \mathrm{proj}^N_k(x, y) = x$

$\mathbb{R}^k$ ——— A x ———→

$\Vert$

$\lim_{N\to\infty} \mathrm{proj}^N_k(\mathbb{S}^N(\sqrt{N-1}))$

Fact. $\ell \in \mathbb{N}_0 := \mathbb{Z}_{\geq 0}$

$$\lambda_\ell(\mathbb{S}^N(\sqrt{N-1})) = \frac{\ell \cdot (\ell + N - 1)}{N - 1}$$

$\downarrow N \to \infty$

$$\lambda_\ell(\Gamma^k) = \ell$$

different multiplicities

$\left(\Delta_{\Gamma^k} f(x) = \Delta f(x) - \langle x, \nabla f(x) \rangle \text{ i.e. } \int_{\mathbb{R}^k} f_1 \Delta_{\Gamma^k} f_2 \, d\gamma^k = -\int_{\mathbb{R}^k} \langle \nabla f_1, \nabla f_2 \rangle d\gamma^k \right)$

Thm. $a_N > 0$ & $\frac{a_N}{\sqrt{N-1}} \xrightarrow{N\to\infty} \gamma > 0$. $\ell \in \mathbb{N}_0$, $k, N \in \mathbb{N}$ w/ $k \le N$

$E_\ell(\mathbb{S}^N(a_N)) := \{\Phi : \ell^{th}$ eigenfunction on $\mathbb{S}^N(a_N)\}$ ∋ homog. harmonic poly on $\mathbb{R}^{N+1}$ (deg $= \ell$)

$E_\ell^k(\mathbb{S}^N(a_N)) := \{Q : \mathbb{R}^k \to \mathbb{R} : \text{poly} \mid Q \circ \mathrm{proj}^N_k|_{\mathbb{S}^N(a_N)} \in E_\ell(\mathbb{S}^N(a_N))\}$

$\downarrow$ $N \to \infty$ strongly in $L^2(\Gamma^k_\gamma)$

$E_\ell(\Gamma^k_\gamma)$ unif on cpt sets $\Gamma^k_\gamma = (\mathbb{R}^k, \gamma^k_\gamma)$ $\gamma^k_\gamma(dx) = (2\pi\gamma^2)^{-\frac{k}{2}} \exp\left(-\frac{|x|^2}{2\gamma^2}\right)$

pf. $Q \in E^k_\ell(\mathbb{S}^N(a_N)) \iff Q = \sum_{L \in \mathbb{N}_0^k(\ell)} a_L Q_{N,k,L}$ $a_L \in \mathbb{R}$

$$\mathbb{N}_0^k(\ell) := \{L \in \mathbb{N}_0^k \mid \sum_{i=1}^k L_i = \ell\}$$

$$Q_{N,k,L}(x) := \sum_{j=0}^{[\frac{\ell}{2}]} (-1)^j \left(\prod_{m=0}^{j} \frac{a_N^2 - |x|^2}{2j(N-1+k-2m}\right) \cdot \left(\sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2}\right)^j \prod_{i=1}^{k} x_i^{L_i}$$

$\downarrow$ $N \to \infty$

$$Q_{k,L}(x) = \sum_{j=0}^{[\frac{\ell}{2}]} (-1)^j \left(\prod_{m=0}^{j} \frac{\gamma^2}{2j}\right)\left(\sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2}\right)^j \prod_{i=1}^{k} x_i^{L_i} \in E_\ell(\Gamma^k_\gamma)$$

※ conv. of heat flow, Cheeger energy / 1st Dirichlet eigen on ball in $\mathbb{S}^N(a_N)$)
half sp in Γ^k_γ