NC convexity and NC function theory

(inspired by Davidson-Kennedy)

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Narrator: (rambles something)

$$\left\{\begin{array}{l}\text{NC topology}\\ \text{NC geometry}\\ \text{NC measure theory}\end{array}\right\} \neq \text{NC function theory}$$

A - commutative C^*-algebra $\cong C(X) \longleftrightarrow X$

equivalence

Thus,

A - nc C^*-algebra $\cong$?? $\longleftrightarrow$ nc topologica space

Dictionary:

weights on vN algebra $\longleftrightarrow$ "nc integration theory"

commuatitive	NC
closed set	ideal
compact	unital
homeomorphism	*-endomorphism
⋮	⋮

$$\underbrace{\text{NC } C^*\text{-algebra}}_{\text{this is interesting}} \cong \underbrace{??}_{\text{what's this??}} \longleftrightarrow \underbrace{\text{“NC topological space”}}_{\text{this is jargon}}$$

Recall: A commutative $\cong C(X)$

Q: $\boxed{\text{How?}}$

<u>Answer:</u> A acts on its irreducible repns

$$\hat{a}(\varphi) = \varphi(a), \quad \varphi \in \mathcal{M}(A)$$

take $X = \mathcal{M}(A)$

<u>NOT A NEW IDEA:</u>

View objects as functions on their representation spaces.

① NC function theory gives* a concrete, useful, and interesting way of viewing operator spaces/algebras as spaces/algebras of "functions" on their "representations".

② What this means precisely?

<u>Many</u> takes.

③ I will present the following take:

K.R. Davidson and M. Kennedy, "Noncommutative Choquet theory". (preprint, 2019)

* - among other things!!

ALL OBJECTS ASSUMED

SEPARABLE** + UNITAL*

*- when it makes sense

** - just don't worry about it,
because it not exactly true (sigh....)

Davidson and Kennedy's framework:

Upcoming slides ← (1) View operator systems as spaces of nc functions on nc convex sets.

(2) View C^*-algebras as algebras of continuous nc functions.

(3) Use this to study operator systems, C^*-algebras, and their interactions.

A theme going back to Arveson, but "algebraically"!

Notation

* $n, m, \ldots$ — cardinal numbers (think $n, m \in \{1, 2, \ldots, \aleph_0\}$)
* E — a dual operator space with weak-$*$ topology (think $\dim E < \infty$)
* $M_{m,n}(E)$, $M_n(E)$

$$\mathcal{M}(E) = \bigsqcup_{n \le \kappa} M_n(E)$$

κ ≡ sufficiently large cardinal

(think $\bigsqcup_n = \bigsqcup_{n \le \aleph_0}$)

Note the difference!!

* **Def:** $K = \bigsqcup K_n \subseteq \mathcal{M}(E)$ is NC CONVEX if it is closed under:
  1. Bdd. direct sums
  2. Compession by isometries in $M_{n,m}$

  1. $\{x_i \in K_i\} \Longrightarrow \sum \alpha_i x_i \alpha_i^* \in K_n$, $\sum \alpha_i \alpha_i^* = 1_n$, $\alpha_i \in M_{n,n_i}$ iso.
  2. $\beta \in M_{n,m}$ iso, $x \in K_n \Longrightarrow \beta^* x \beta \in K_m$

2 FACTS: $K \subseteq M(E)$, E dual operator space

① NC CONVEX $\Longleftrightarrow$ Closed under NC convex combinations

we need this to converge $\longleftarrow$ $\sum \alpha_i^* x_i \alpha_i \in K_n$

when $x_i \in K_{n_i}$, $\alpha_i \in M_{n_i,n}$

$\sum \alpha_i^* \alpha_i = 1_n$

② Levelwise closed $\Longrightarrow$ "Globally closed"

$\underbrace{\alpha_i \in M_{n,n_i},\ \alpha_i^* \alpha_i = 1_{n_i},\ \alpha_i \alpha_i^* \longrightarrow 1_n,\ x_i \in K_{n_i}}$

$\lim \alpha_i x_i \alpha_i^* = x \in M_n(E)$

$\Longrightarrow x \in K_n$

see why we needed weak-*?

Matrix convex	NC convex
⊛ $\bigsqcup_{n=1}^{\infty} K_n = \bigsqcup_{n<\infty} K_n$	⊛ $\bigsqcup_{n\le\aleph_0} K_n$ or other cardinal
⊛ Wittstock 1981 (Effros-Winkler)	⊛ Davidson-Kennedy 2019
⊛ Closed under (finite) matrix convex combinations	⊛ closed under (infinite) matrix convex combinations
⊛ $\underbrace{\mathrm{Ext}(K)}$ might be empty depends on definition (Webster-Winkler)	⊛ Ext(K) generates K coming up!
Determines uniquely a nc cnv set $\bigsqcup_{n\le\aleph_0} K_n$	Determined uniquely by finite part an vice versa

dual equivalence with operator systems

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<u>Example:</u> S operator system.

NC state space $K = \bigsqcup_n K_n$

$$K_n = UCP(S, M_n) \subseteq CB(S, M_n) = M_n(\underbrace{S^*}_{E = S^*})$$

☺

K "reprn's" of S on $M = \bigsqcup_n M_n$

$\Longrightarrow$ S functions on K: $\hat{a}(\varphi) := \varphi(a) \in M_n$

$(a \in S,\ \varphi \in K_n)$ graded

<u>Note:</u> ① $\hat{a}$ is <u>continuous</u>:

$$\varphi_n \longrightarrow \varphi \implies \hat{a}(\varphi_n) = \underset{\text{(by def.)}}{\varphi_n(a)} \xrightarrow{\text{weak-*}} \varphi(a)$$

② $\hat{a}$ is <u>nc affine</u>:

$$\gamma_i \in M_{n_i,n} \quad \sum \gamma_i^* \gamma_i = 1_n, \quad X_i \in K_{n_i} \implies \hat{a}\left(\sum \alpha_i^* X_i \alpha_i\right) = \sum \alpha_i^* \hat{a}(X_i) \alpha_i$$

Def: K nc convex set.

$A(K)$ — space of continuous nc affine maps

$a : K = \bigsqcup_n K_n \longrightarrow M = \bigsqcup_n M_n$

Rem: $A(K)$ is an operator system.

Example: $a \in M_n(A(K)) = A(K, M_n)$ is ≥ 0

$\Longleftrightarrow$

$M_n(M_m) \ni a(x) \geq 0$ for all $x \in K$

if $x \in K_m$

Thm: K (cpt nc-cvx over $E = (E_x)^*$) $\underset{\text{nc affine homeo.}}{\cong}$ NC state space of $A(K)$ $UCP(A(K), M)$

Thm: S op. system, K nc state space $\underset{\text{complete order iso.}}{\cong}$ $A(K)$ $[a \mapsto \hat{a}]$

functions!!

Recap

op. system $=^*$ continuous $\boxed{\text{nc}}$ affine functions on $\boxed{\text{nc}}$ cnvx + cpt. sets

$$S \rightsquigarrow \coprod_n UCP(S, M_n)$$

nc state space

$$A(K) \leftsquigarrow K$$

* Categorigal duality, UCP maps $\longleftrightarrow$ nc affine maps

Question

$$\begin{tikzcd} \text{Operator system } S \arrow[r, leftrightarrow] \arrow[d, hook] & \text{continuous nc affine functions on a compact nc convex set} \arrow[d, hook] \\ C^*\text{-algebra (say, } C^*(S)) \arrow[r, leftrightarrow] & ?? \end{tikzcd}$$

Important aspect of noncommutative analysis. (Arveson)

$\boxed{\text{* How is a } C^*\text{-algebra viewed as an algebra of functions??}}$

"Top-down" answer

Takesaki and **Bichteler's** noncommutative Gelfand theory:

<u>Def:</u> $T : \mathrm{Rep}(A, H) \longrightarrow B(H)$ is an <u>admissible operator field</u> if:

1. $\|T\| := \sup\{\|T(\pi)\| : \pi \in \mathrm{Rep}(A,H)\} < \infty$
2. $T(\pi_1 \oplus \pi_2) = T(\pi_1) \oplus T(\pi_2)$ ← respects direct sums
3. $T(\pi^u) = u^* T(\pi) u$ ← respects "unitary equivalence"

<u>Thm:</u> ① $A^{**} \cong$ The C^*-algebra of all admissible operator fields

via $b \longmapsto \hat{b}$ $\qquad \hat{b}(\pi) = \pi(b)$

② $A =$ The C^*-subalgebra of <u>continuous</u> admissible fields.

point-~~[illegible]~~strong to strong-*

Rmk: Takesaki used this to prove Takeda's Thm: $A^{**} \cong \pi_u(A)''$

"Bottom-up" answer

Def: Let K be a compact nc convex set. $f: K \to \mathcal{M}$ is an nc function if

1. $f(K_n) \subseteq \mathcal{M}_n$ (graded)
2. $f\left(\sum \alpha_i x_i \alpha_i^*\right) = \sum \alpha_i f(x_i) \alpha_i^*$ (respects direct sums)
   $x_i \in K_{n_i}$, $\alpha_i \in \mathrm{Isom}(\mathcal{M}_{n_i,n})$, $\sum \alpha_i \alpha_i^* = 1_n$
3. $f(\beta^* x \beta) = \beta^* f(x) \beta$, for $x \in K_n$, $\beta \in \mathcal{U}(\mathcal{M}_n)$ (respects unitary conjugation)

1. Change unitary to isometry $\Rightarrow$ nc affine
2. Change unitary to invertible, $\beta^* \rightsquigarrow \beta^{-1}$ nc analytic functions
3. Also $\beta \in$ Group

Notation:

$$B(K) = \left\{ f: K \xrightarrow{nc} \mathcal{M} \;\middle|\; \|f\|_\infty := \sup_{x \in K} \|f(x)\| < \infty \right\}$$

$B(K)$ is a C^*-algebra w/ pt.-wise operations.

$$C(K) := C^*(A(K)) \subseteq B(K)$$

Theorem : $B(K) \cong C^*_{max}(A(K))^{**} \cong$ Bdd. + Admissible operator fields

$C^*(A(K))$ in $B(K)$ $=: C(K) \cong C^*_{max}(A(K)) \cong$ ultrastrong $*$-continuous bdd. + admissible operator fields

(Thm!!)

ultrastrong $*$-continuous nc functions

topology....

Kirchberg-Wasserman

C^* max: For every operator system S $\exists\, u: S \xrightarrow[ucis]{} C^*_{max}(S) = C^*(u(S))$ s.t.:

$$\begin{tikzcd} & C^*_{max}(S) \arrow[d, dashed, two heads, "\pi"] \\ S \arrow[ur, "u"] \arrow[r, "v"', "ucis"] & C^*(v(S)) \end{tikzcd}$$

Maximal and minimal C^*-algs.

$$A(K) = S \xrightarrow[\text{ucis}]{u} C^*_{max}(S) = C(K)$$

$$C^*_{max}(S) \xrightarrow{\exists!\,\pi \ *\text{-hom.}} C^*(v(S))$$

$$A(K) = S \xrightarrow[\text{ucis}]{v} C^*(v(S))$$

$$C^*(v(S)) \xrightarrow{\exists!\,\rho \ *\text{-hom.}} C^*_{min}(S)$$

$$A(K) = S \xrightarrow{i} C^*_{min}(S) = C^*_e(S) = C(\partial_{\kappa} K)$$

later

Commutative picture:

(analogy!)

C compact convex set

$$A(C) \hookrightarrow C(C) = C^*(A(C))$$

SW thm.

$$C(C) \xrightarrow{R} C(\partial_s C)$$

$$A(C) \hookrightarrow C(\partial_s C)$$

$$\partial_s C = \overline{ext(C)}$$

Coming up:

Clarify roles of $\boxed{C^*_{max}}(A(K))$ and $\boxed{C^*_{min}}(A(K))$

C^*_{max}

If $\underset{\substack{\| \\ UCP(A(K), M_n)}}{x \in K_n} \subseteq K \overset{\text{univ. prop.}}{\Longrightarrow}$ x etends to $\delta_x \in Rep(C^*_{max}, M_n)$

– Conversely –

If $\pi \in Rep(C^*_{max}, M_n) \Longrightarrow \pi$ restricts to $x = \pi|_{A(K)} \in \underset{\substack{\| \\ UCP(A(K), M_n)}}{K_n}$

$\Rightarrow \pi = \delta_x$

K = Rep'ns of C^*_{max} = evaluations at points

$\Rightarrow$ UCP maps $C^*_{max} \to M$ = "probability measures"

C^*_{min}

The set of nc extreme points $\partial K \subseteq K$ is defined in such a way, such that,

$$C^*_{min}(A(K)) \cong C(K)\big|_{\partial K} \overset{?}{\cong} C(\overline{\partial K})$$

The extreme points $x \in \partial K$ are precisely the $x \in K$ s.t. δ_x is a boundary representation.

∂K — the noncommutative Choquet boundary

we'll try to make sense of this

Pure, maximal, extreme

Def: $y \in K_n$ is a dilation of $x \in K_m$ if $\exists \alpha \in \mathrm{Iso}(M_{n,m})$

$$x = \alpha^* y \alpha$$

x is maximal if the only dilations are trivial $y = x \oplus z$

Dritschel-McCullough ← **Thm:** Every point in a compact nc convex set K has a maximal dilation.

prop. (but easy) ← **Def:**/~~prop~~: A point $x \in K$ is said to be extreme if it is maximal and pure:

finite nc convex combination $\left\{ \begin{array}{l} x = \sum \alpha_i^* x_i \alpha_i \\ \sum \alpha_i^* \alpha_i \end{array} \right\} \Longrightarrow \begin{array}{l} \alpha_i = c_i \beta_i, \quad c_i \geq 0, \beta_i \text{ iso} \\ \beta_i^* x_i \beta_i = x \end{array}$

Rmk: If x is pure $\Longrightarrow \delta_x : C(K) \to M_n$ is irreducible

Prop: x is extreme $\iff \delta_x : C(K) \to M_n$ is a boundary rep.

$\iff$ $\boxed{\delta_x \text{ is irreducible + is the unique representing map for } x.}$

Krein - Milman Theorems

Krein-Milman

Thm: A compact nc convex set is the closed nc convex hull of its extreme points.

About proof: Assume $\overline{\mathrm{ncconv}}(\partial K) \subsetneq K$

Separate $x_0 \in K \setminus \overline{\mathrm{ncconv}}(\partial K)$ from $\overline{\mathrm{ncconv}}(\partial K)$ with nc affine function via nc Hahn Banach thm → contradiction

Crucial: ∂K completely norms $A(K)$.

How?? Sufficiently many bdry reps.
(Arveson, Davidson - Kennedy x2)

"Milman's converse"

Thm: K cpt. nc convex set, $X \subseteq K$ closed and closed under compressions. ($\alpha^* X_n \alpha \subseteq X_m$, $\forall \alpha \in \mathrm{iso}(M_{n,m})$)

$$\overline{\mathrm{ncconv}}(X) = K \implies \partial K \subseteq X$$

Example: commutative case

* $C \subseteq E$ compact and convex $\quad A(C) \subseteq C(C)$

(continuous affine) $\quad$ (continuous)

$\to$ not $C^*_{max}(A(K))$

defines operator system structure $A(C) \cong A(K)$

* K nc state space of $A(C)$

* $C^*_{min}(A(K)) = C(\overline{\mathrm{ext}\, C})$

* $\partial K = \mathrm{ext}(C)$:

$x \in \partial K \Rightarrow$

δ_x maximal, irreducible, factors through C^*_{min}

$\Rightarrow x \in K_1 \cong C$ and must be extreme

$\Rightarrow x \in \mathrm{ext}(C)$

$x \in \mathrm{ext}(C)$

$\Rightarrow$ maximal dilation $x = \alpha^* y \alpha$

$\Rightarrow$ define $\mu : C(K) \to \mathbb{C}$, $\mu = \alpha^* \delta_y \alpha$

$\Rightarrow \delta_y$ factors through C^*_{min}

$\mu : C(\overline{\mathrm{ext}\, C}) \to \mathbb{C}$

it's a measure on $\overline{\mathrm{ext}(C)}$

x extreme

$\Rightarrow \mu = \delta_x \Rightarrow$ maximal + irreducible $\in \partial K$

Toy application + example

Irreducible representations of the rational rotation algebras

* $\theta = 2\pi \frac{m}{n}$, $\gcd(m,n) = 1$
* $A_\theta = C^*(u,v \mid vu = e^{i\theta} uv)$
* $X = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & & 1 & \\ & & & \ddots & \\ 1 & & & & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & & & \\ & e^{i\theta} & & \\ & & \ddots & \\ & & & e^{i(n-1)\theta} \end{bmatrix}$

$u \mapsto X$, $v \mapsto Y$ determines irreducible representation, of A_θ.

* Every irrep. has the form* $u \mapsto \alpha X$, $v \mapsto \beta Y$ $\quad \alpha, \beta \in \mathbb{T}$ (well known, but the proof we give generalizes)

* up to unitary equivalence

Proof:

why should every irrep. be a summand?

1. construct $(U, V) = \bigoplus_{\alpha,\beta\in\mathbb{T}} (\alpha X, \beta Y)$

2. This defines faithful rep. of A_θ (gauge invariance $\mathbb{T}^2$)

3. consider $A = \{(\alpha X, \beta Y) \mid \alpha,\beta\in\mathbb{T}\} \subseteq W(u,v) \cong$ NC state space of $OS(u,v)$

4. $B =$ clouser of compressions of A

5. Bdry reps. of $OS(u,v)$ in $A_\theta \longleftrightarrow \partial W(u,v)$ ($\delta_x \longleftrightarrow x$)

$\quad ||$

all irreducible reps. (unitaries are hyperrigid)

6. $\overline{\mathrm{nc\,conv}}(B) = W(u,v)$

7. Milman's converse $\Longrightarrow \partial W(u,v) \subseteq B$

all irreps.

contains unitary conjugates of $(\alpha X, \beta X)$

$\boxed{\text{only possibilities}}$

Thank you, organizers.

Thank you, audience.