

NC  
convexity  
and

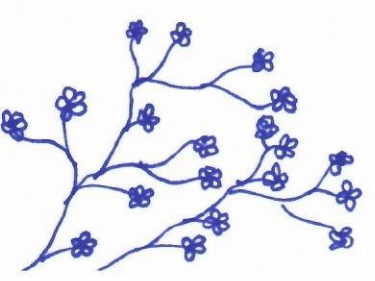
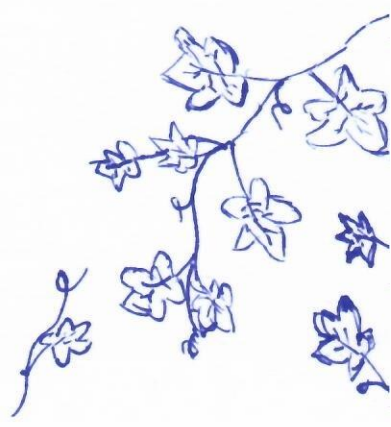
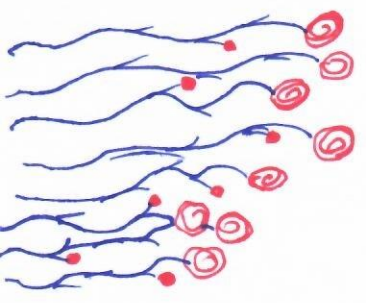
NC  
function theory

(inspired by Davidson-Kennedy)

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Or Shalit, Technion

BIRS - CMO, August 2021



Narrator: (rambles something)

$\left\{ \begin{array}{l} NC \\ NC \\ NC \end{array} \right\} \begin{array}{l} \text{topology} \\ \text{geometry} \\ \text{measure theory} \end{array} \neq NC \text{ function theory}$

$A$  - commutative  
 $C^*$ -algebra

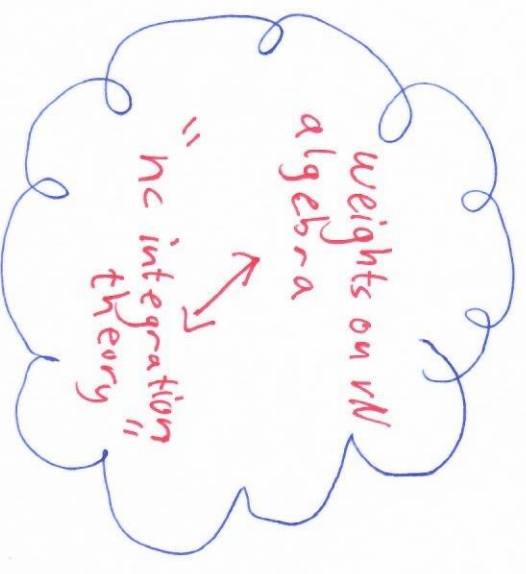
$$\cong C(X) \longleftrightarrow X$$

↙  
 equivalence

Thus,

$$A\text{-}nc \ C^*\text{-algebra} \cong ?? \longleftrightarrow nc \ topology \ space$$

Dictionary:



Commutative

$Nc$

closed set

compact

homeomorphism

- o
- o
- o

ideal

unital

\*-endomorphism

- o
- o
- o

$NC$   $C^*$ -algebra

this is interesting

$\cong$  ???

$\longleftrightarrow$   $NC$  "topological space"

this is jargon

what's this??

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Recall:  $A$  commutative  $\cong \mathbb{C}(X)$

Q: How?

Answer:  $A$  acts on its irreducible reps

$\hat{\chi}(\varphi) = \varphi(a)$ ,  $\varphi \in \mathcal{M}(A)$   
take  $X = \mathcal{M}(A)$

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NOT A NEW IDEA:

View objects as functions on their representation spaces.

①  $NC$  function theory gives\* a concrete, useful, and interesting way of viewing operator spaces/algebras as ~~a~~ spaces/algebras of "functions" on their "representations".

② What this means precisely?

Many takes.

③ I will present the following take:

K.R. Davidson and M. Kennedy, "Noncommutative Choquet theory."  
(preprint, 2019)

\* - among other things!!

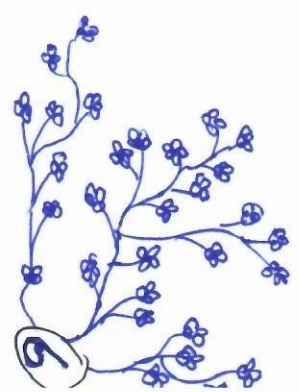


ALL OBJECTS ASSUMED  
SEPARABLE\*\* + UNITAL\*



\* - when it makes sense

\*\* - just don't worry about it,  
because it not exactly true (sigh....)



# Davidson and Kennedy's framework:

Upcoming slides

← ① View operator systems as spaces of  $n_c$  functions on  $n_c$  convex sets.

② View  $C^*$ -algebras as algebras of continuous  $n_c$  functions.

③ Use this to study operator systems,  $C^*$ -algebras, and their interactions.

A theme going back to Arveson, but "algebraically".



# Notation

\*  $n, m, \dots$  — cardinal numbers (think  $n, m \in \{1, 2, \dots, \aleph_0\}$ ) (think  $\dim E < \infty$ )

\*  $E$  — a dual operator space with weak-\* topology

\*  $M_{m,n}(E), M_n(E)$

$$M(E) = \bigsqcup_{n \leq \aleph_0} M_n(E)$$

sufficiently large cardinal

$$\left( \text{think } \bigsqcup_n = \bigsqcup_{n \leq \aleph_0} \right)$$

Note the difference!!

\* Def:  $K = \bigsqcup K_n \subseteq M(E)$  is **NC CONVEX** if

it is closed under:

- ① Bdd. direct sums
  - ② Compression by isometries in  $M_{n,m}$
- ①  $\{X_i \in K_i\} \xRightarrow{\text{iso.}} \sum \alpha_i X_i \alpha_i^* \in K_n$   
 $\sum \alpha_i \alpha_i^* = I_n$
- ②  $B \in M_{n,m}^{\text{iso}}, X \in K_n \xRightarrow{\text{iso.}} B^* X B \in K_m$

2 FACTS:  $K \subseteq M_n(E)$ ,  $E$  dual operator space

① NC CONVEX  $\Leftrightarrow$  Closed under

we need this to converge

NC CONVEX combinations  
 $\leftarrow \sum \alpha_i^* x_i \alpha_i \in K_n$   
 when  $x_i \in K_{n_i}$ ,  $\alpha_i \in M_{n_i, n}$   
 $\sum \alpha_i^* \alpha_i = I_n$

② Levelwise closed

$\Rightarrow$  " Globally closed "

see why we needed weak-\*?

$\alpha_i \in M_{n_i, n_i}$ ,  $\alpha_i^* \alpha_i = I_{n_i}$   
 $\alpha_i \alpha_i^* \rightarrow I_n$ ,  $x_i \in K_{n_i}$   
 $\lim \alpha_i x_i \alpha_i^* = X \in M_n(E)$   
 $\Rightarrow X \in K_n$

# Matrix convex

\*  $\bigsqcup_{n=1}^{\infty} K_n = \bigsqcup_{n < \infty} K_n$

\* Wittstock 1981  
(Effros-Winkler)

\* Closed under (finite) matrix convex combinations

\*  $\text{Ext}(K)$  might be empty

depends on definition

Determines uniquely a nc cv set  $\bigsqcup_{n \leq X_0} K_n$

# NC convex

\*  $\bigsqcup_{n \leq X_0} K_n$

or other cardinal 

\* Davidson - Kennedy 2019

\* closed under (infinite) matrix convex combinations

\*  $\text{Ext}(K)$  generates  $K$

coming up!

Determined uniquely by finite part

dual equivalence of operator systems

(Webster-Winkler)

Example:  $S$  operator system.

$N \subset$  state space  $K = \bigsqcup_n K_n$

$K_n = \text{UCP}(S, M_n) \subseteq \text{CB}(S, M_n) = M_n(S^*)$   
 $E = S^*$

$K$  "repr's" of  $S$  on  $M = \bigsqcup_n M_n$

$\implies S$  functions on  $K$ :  $\hat{\alpha}(\varphi) := \varphi(a) \in M_n$



Note: ①  $\hat{\alpha}$  is continuous:

$\varphi_n \rightarrow \varphi \implies \hat{\alpha}(\varphi_n) = \varphi_n(a) \xrightarrow{\text{weak}^*} \varphi(a)$   
(by def)

②  $\hat{\alpha}$  is  $N \subset$  affine:

$\chi_i \in M_{n_i n}$   $\sum \alpha_i^* \chi_i = I_n, \chi_i \in K_{n_i} \implies \hat{\alpha}(\sum \alpha_i^* \chi_i \alpha) = \sum \alpha_i^* \hat{\alpha}(\chi_i) \alpha$

⑩

Def:  $K$  nc convex set.

$A(K)$  — space of continuous nc affine maps  
 $\alpha: K = \bigcup_n K_n \longrightarrow M = \bigcup_n M_n$

Rem:  $A(K)$  is an operator system.

Example:  $\alpha \in M_n(A(K)) = A(K, M_n)$  is  $\succeq 0$



$M_n(M_m) \ni \alpha(x) \succeq 0$  for all  $x \in K$   
if  $x \in K_m$

Thm:  $K \cong \begin{matrix} \cong \\ \text{nc affine} \\ \text{homos.} \end{matrix} \begin{matrix} \text{NC} \\ \text{state space} \end{matrix} \cong \begin{matrix} \text{UCP}(A(K), M) \end{matrix}$   
(cpt nc-conv over  $E = (E_x)^*$ )

Thm: S op. system,  
 $K$  nc state space

$\cong$   
complete order iso.

$A(K) \xrightarrow{a \mapsto \hat{a}}$   
functions!!  
(II)

# Recap

op. system  $\equiv^*$

Continuous  $\boxed{nc}$  affine  
functions on  $\boxed{nc}$  conv  
cpt. sets

$S \rightsquigarrow \bigsqcup_n \text{UCP}(S, M_n)$

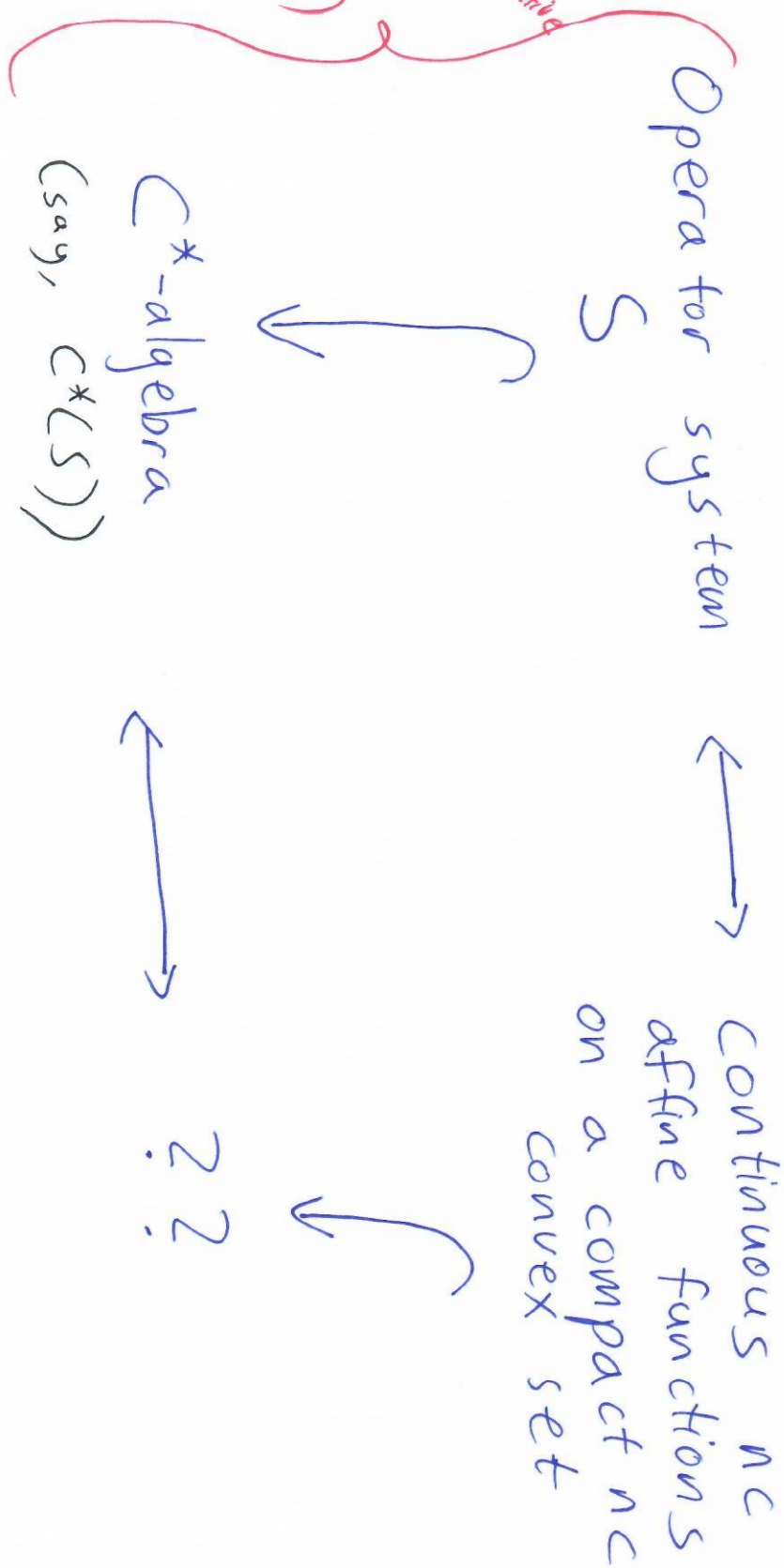
$nc$  state space  $\leftarrow$

$A(K) \rightsquigarrow K$

\* Categorical duality, UCP maps  $\leftarrow$   $nc$  affine maps

# Question

Important aspect of noncommutative analysis.  
(Arveson)



\* How is a  $C^*$ -algebra viewed as an algebra of functions??

"Top-down" answer

**Takesaki** and **Bichteler's** noncommutative Gelfand theory:

Def:  $T: \text{Rep}(A, H) \rightarrow B(H)$  is an admissible operator

field if:  $\textcircled{1} \|T\| := \sup \{ \|T\pi\| : \pi \in \text{Rep}(A, H) \} < \infty$

respects direct sums  $\leftarrow \textcircled{2} T(\pi_1 \oplus \pi_2) = T(\pi_1) \oplus T(\pi_2)$

respects unitary equivalence  $\leftarrow \textcircled{3} T(\pi^u) = U^* T(\pi) U$

Thm:  $\textcircled{1} A^{**} \cong$  The  $C^*$ -algebra of all admissible operator fields

$\textcircled{2} A \xrightarrow{\text{via}} \hat{b} \xrightarrow{\hat{b}(\pi) = \pi(b)}$  The  $C^*$ -subalgebra of continuous admissible fields.

point-to string\*

Rmk: Takesaki used this to prove Takesaki's Thm:  $A^{**} \cong \pi_u(A)$



"Bottom-up" answer

Def: Let  $K$  be a compact nc convex set.

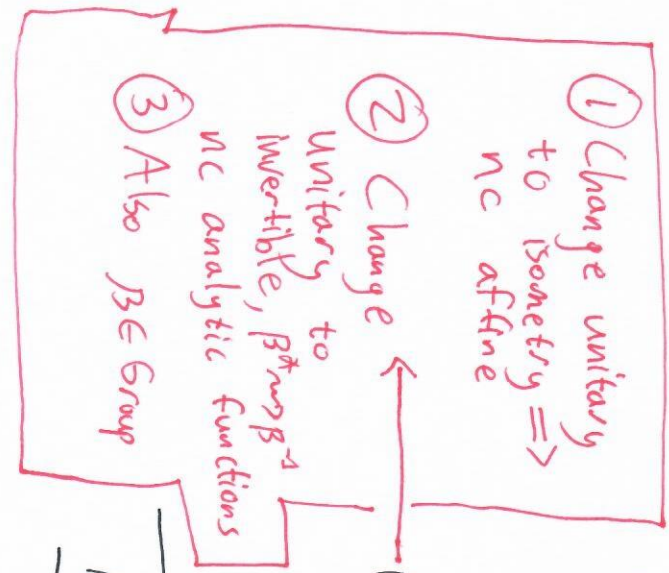
$f: K \rightarrow M$  is an nc function if

①  $f(K_n) \subseteq M_n$  (graded)

②  $f(\sum \alpha_i X_i \alpha_i^*) = \sum \alpha_i f(X_i) \alpha_i^*$  (respects direct sums)

$X_i \in K_{n_i}$ ,  $\alpha_i \in \text{Iso}_n(M_{n_i, n})$ ,  $\sum \alpha_i \alpha_i^* = I_n$

③  $f(B^* X B) = B^* f(X) B$ , for  $X \in K_n$ ,  $B \in \mathcal{U}(M_n)$  (respects unitary conjugation)



Notation:

$B(K) = \{ f: K \xrightarrow{nc} M \mid \|f\|_\infty := \sup_{X \in K} \|f(X)\| < \infty \}$

$B(K)$  is a  $C^*$ -algebra w/ pt.-wise operations.

$C(K) := C^*(A(K)) \subseteq B(K)$

Theorem:  $B(K) \cong \cong \cong C_{\max}^*(A(K))^{**} \cong$  Admissible operator fields

$$C^*(A(K)) \text{ in } B(K) \cong C(K) \cong \cong C_{\max}^*(A(K)) \cong \cong \text{ultrastrong } * \text{-continuous bdd. radmissible operator fields}$$

↙ (Thm!!)



ultrastrong \* - continuous n.c. functions

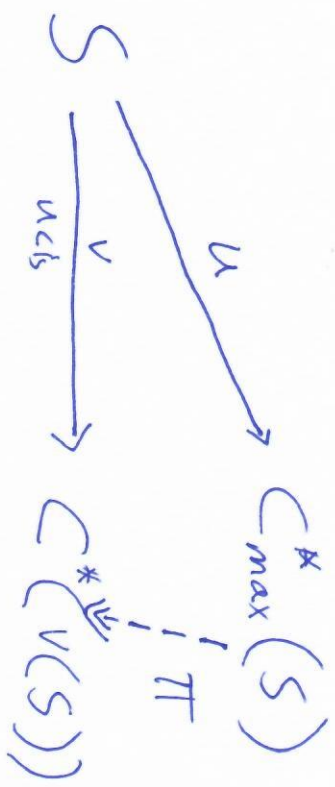


topology ...

Kirchberg-Wasserman

$C_{\max}^*$  For every operator system  $S$

$$\exists u: S \xrightarrow{u} C_{\max}^*(S) = C^*(u(S)) \text{ s.t. :}$$



Maximal and minimal  $C^*$ -algs.

$$C_{\max}^*(S) = C(K)$$

$\downarrow \exists! \pi$  \*-hom.

$$A(K) = S \xrightarrow{V} C^*(V(S))$$

$\downarrow \exists! \rho$  \*-hom.

$$C_{\min}^*(S) = C_e^*(S) = C(\partial_e K)$$

later  $\leftarrow$

Commutative picture:

$C$  compact convex set

(analogy!)

$$A(C) \xrightarrow{\quad} C(C) \xrightarrow{\quad} C^*(A(C))$$

$\downarrow$  SW thm.

$$C(C) \xrightarrow{\quad} C(\partial_e C)$$

$\parallel$  ext(C)

Coming up:

Clarify roles of  $C_{\max}^*(A(K))$  and  $C_{\min}^*(A(K))$

$C_{\max}^*$

*univ. prop.*

$$\begin{aligned} \text{If } X \in K_n \subseteq K &\implies X \text{ extends to } \int_X \in \text{Rep}(C_{\max}^*, M_n) \\ &\stackrel{\parallel}{=} \text{UCP}(AK), M_n \end{aligned}$$

- Conversely -

$$\text{If } \pi \in \text{Rep}(C_{\max}^*, M_n) \implies \pi \text{ restricts to } X = \pi|_{AK} \in K_n$$

$\parallel$   
 $\text{UCP}(AK), M_n$

$$\implies \pi = \int_X$$

---

$K = \text{Repins of } C_{\max}^* = \text{evaluations at points}$

$\implies \text{UCP maps } C_{\max}^* \rightarrow M = \text{"probability measures"}$

$C_{\min}^*$

The set of nc extreme points  $\partial K \subseteq K$ ,  
is defined in such a way, such that,

$$C_{\min}^*(A(K)) \cong C(K) \Big|_{\partial K} \stackrel{?}{\cong} C(\overline{\partial K})$$

The extreme points  $x \in \partial K$  are precisely the  
 $x \in K$  s.t.  $\delta_x$  is a boundary representation.

$\partial K$  — the noncommutative Choquet boundary

we'll try to make sense of this

Pure, maximal, extreme

Def:  $y \in K_n$  is a dilation of  $x \in K_m$  if  $\exists \alpha \in \text{Iso}(M_n, m)$

$$X = \alpha^* y \alpha$$

$X$  is maximal if the only dilations are trivial  $y = X \oplus Z$

*Dritschel-McCullough*

Thm: Every point in a compact nc convex set  $K$  has a maximal dilation.

*prop. (but easy)*

Def: ~~prop.~~ A point  $x \in K$  is said to be extreme if it is maximal and pure:

$$\left\{ \begin{array}{l} \text{finite nc} \\ \text{convex combination} \end{array} \right\} \left\{ \begin{array}{l} X = \sum \alpha_i^* X_i \alpha \\ \sum \alpha_i^* \alpha_i \end{array} \right\} \implies \left\{ \begin{array}{l} \alpha_i = c_i B_i, \quad c_i \geq 0, B_i \text{ iso} \\ B_i^* X_i B_i = X \end{array} \right.$$

Rmk: If  $X$  is pure  $\implies \delta_X : C(K) \rightarrow M_n$  is irreducible

Prop:  $X$  is extreme  $\iff \delta_X : C(K) \rightarrow M_n$  is a boundary rep.

$\iff \delta_X$  is irreducible + is the unique representing map for  $X$ .

# Krein - Milman Theorems

Krein - Milman

Thm: A compact  $n_c$  convex set is the closed  $n_c$  convex hull of its extreme points.

About proof: Assume  $\overline{n_c \text{conv}(\partial K)} \neq K$

Separate  $X_0 \in K \setminus \overline{n_c \text{conv}(\partial K)}$  from  $\overline{n_c \text{conv}(\partial K)}$  with  $n_c$  affine function via  $n_c$  Hahn Banach thm

control choices

Crucial:  $\partial K$  completely norms,  $A(K)$ .

How?? Sufficiently many bdry reps. (Arveson, Davidson - Kennedy X2)

"Milman's converse"

Thm:  $K$   $n_c$  cpt.  $n_c$  convex set,  $X \subseteq K$  closed and closed under compressions.  $(\alpha^* X_n \alpha \subseteq X_m, \text{Arviso } (M_{n,m}))$

$$\overline{n_c \text{conv}(X)} = K \implies \partial K \subseteq X$$

# Example: commutative case

\*  $C \subseteq E$   
compact and convex



\*  $K$  nc state space of  $A(C)$

\*  $C_{\min}^*(A(K)) = C(\overline{\text{ext}(C)})$

\*  $\partial K = \text{ext}(C)$  ;

$X \in \partial K \implies$

$\delta_X$  maximal, irreducible, factors through  $C_{\min}^*$

$\implies X \in K_1, \overline{\Delta} \subset C$  and must be extreme

$\implies X \in \text{ext}(C)$

$X \in \text{ext}(C)$

$\implies$  maximal dilation  $X = \alpha^* y \alpha$

$\implies$  define  $\mu: C(K) \rightarrow \mathcal{A}$   
 $\mu = \alpha^* \delta_y \alpha$

$\implies \delta_y$  factors through  $C_{\min}^*$   
 $\mu: C(\overline{\text{ext}(C)}) \rightarrow \mathcal{A}$

$X$  extreme  $\leftarrow$   
 $\implies \mu = \delta_X \implies$  maximal + irreducible  $\in \partial K$



Toy application + example  
 Irreducible representations of  
 the rational rotation algebras



\*  $\Theta = 2\pi \frac{m}{n}$ ,  $\gcd(m, n) = 1$

\*  $A_\Theta = C^*(u, v \mid vu = e^{i\Theta} uv)$

\*  $X = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & & 1 \\ 1 & & & \end{bmatrix}$ ,  $Y = \begin{bmatrix} 1 & & & \\ & e^{i\Theta} & & \\ & & \ddots & \\ & & & e^{i(n-1)\Theta} \end{bmatrix}$

$u \mapsto X$ ,  $v \mapsto Y$  determines  
 irreducible representation, of  $A_\Theta$ .

\* Every irrep. has the form\*

$u \mapsto \alpha X$ ,  $v \mapsto \beta Y$   
 $\alpha, \beta \in \mathbb{T}$

\* up to unitary equivalence

(well known, but the proof we give generalizes)

Proof:

① Construct  $(U, V) = \bigoplus_{\alpha, \beta \in \mathbb{T}} (\alpha X, \beta Y)$

why should every irrep be a subrepresentation?

② This defines faithful rep. of  $A_\theta$  (gauge invariance)

③ consider  $A = \{(\alpha X, \beta Y) \mid \alpha, \beta \in \mathbb{T}\} \subseteq W(u, v) \cong$  NC state space of  $OS(u, v)$

④  $B =$  closure of compressions of  $A$

⑤ Bdry reps. of ~~OS~~  $OS(u, v)$  in  $A_\theta \xleftrightarrow{\delta_x} \rightarrow \partial W(u, v)$   
 $\parallel$   
 all irreducible reps. (unitaries are hyperinvariant)

⑥  $\overline{nc\text{-}cov}(B) = W(u, v)$

⑦ Milman's converse  $\implies \partial W(u, v) \subseteq B$

all irreps.

contains unitary conjugates of  $(\alpha X, \beta X)$

only possibilities

Thank you, organizers.

Thank you, audience.

