Geometry of free loci and factorization of nc polynomials

(following Helton, Klep, Volčič)

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Hilbert's nullstellensatz:
If f_1, f_2 are polynomials over $\mathbb{C}$, and $f_2(x) = 0$ whenever $f_1(x) = 0$, then some power of f_2 belongs to the ideal generated by f_1.
Special cases:

- If f_1 is irreducible, then $f_2 = gf_1$ for some polynomial g.
- If f_1 and f_2 are irreducible and have the same zeroes, then $f_1 = cf_2$ for a nonzero constant $c \in \mathbb{C}$.

$\mathbb{C}\langle x_1, \ldots, x_g \rangle$: polynomials in g noncommuting variables

$f(x_1, x_2) = x_1 x_2 - x_2 x_1$

$f(x_1, x_2, x_3) = 1 + 2x_1 x_2 - x_2 x_1 + x_3 x_2^2 + 7x_2 x_3 x_2$

What should we mean by a "zero" of an nc polynomial?....where do we even evaluate them?
On matrices, of arbitrary size....

Let f be an nc polynomial in g variables $x_1, \ldots, x_g$. For each $n \geq 1$ we get a function

$$f_n : \mathbb{M}_n(\mathbb{C})^g \to \mathbb{M}_n$$

$$f_n : (X_1, \ldots, X_g) \to f(X_1, \ldots, X_g)$$

(a "graded function")

Can also consider polynomials with matrix coefficients:

$g(x_1, x_2) = Ax_1x_2 - Bx_2x_1$

where $A, B \in \mathbb{M}_d$.

Evaluate on $n \times n$ X's:

$$g(X_1, X_2) = A \otimes X_1X_2 - B \otimes X_2X_1 \in \mathbb{M}_d \otimes \mathbb{M}_n \cong \mathbb{M}_{dn}$$

Zeroes of nc polynomials:

- "hard" zeroes: say $X = (X_1, \dots, X_g)$ is a **hard zero** of f if

$$f(X_1, \dots, X_g) = 0_n.$$

Question about hard zeroes[1]: if f_1, f_2 are nc polynomials and f_2 has a hard zero everywhere f_1 does, how are they related?

$$f_1(x,y) = \text{anything}$$
$$f_2(x,y) = xy - yx$$

every hard zero of f_1 **at level** 1 is a hard zero of f_2....

[1] **Hartnullstellenfrage**

Theorem (Amitsur 1957)

Fix a level n. If $f_2(x) = 0_n$ whenever $f_1(x) = 0_n$, then f_2 belongs to the ideal generated by f_1 and $\mathfrak{M}_n$.

$\mathfrak{M}_n$ = ideal of polynomials that are **identically** zero up to level n

Zeroes of nc polynomials:

- "hard" zeroes: $f(X) = 0_n$
- "detailed" zeroes: say a pair $X \in \mathbb{M}_n(\mathbb{C})^g$, $0 \neq v \in \mathbb{C}^n$ is a **detailed zero** of f if

$$f(X_1, \dots, X_g)v = 0.$$

Question about detailed zeroes[2]:

if $f_2(X)v = 0$ whenever $f_1(X)v = 0$, how are f_1, f_2 related?

[2] **Ausführlichnullstellenfrage**

Theorem (Bergman's nullstellensatz (Helton-McCullough 2004))

If $f_2(X)v = 0$ whenever $f_1(X)v = 0$, then f_2 belongs to the left ideal generated by f_1.

(need only check (X, v) up to some fixed size depending on degrees of f_1, f_2)

Zeroes of nc polynomials:

- "hard" zeroes: $f(X) = 0_n$
- "detailed" zeroes: $f(X)v = 0$
- the "zero locus":

$$\mathscr{Z}_n(f) = \{X \in \mathbb{M}_n(\mathbb{C})^g : \det f(X) = 0\}$$

$$\mathscr{Z}(f) = \bigcup_{n \geq 1} \mathscr{Z}_n(f)$$

(Question about the zero locus, etc...) Example:

$$f_1(x, y) = 1 - xy, \quad f_2(x, y) = 1 - yx$$

$$\det(1 - xy) = 0 \quad \text{iff} \quad \det(1 - yx) = 0 :$$

Proof 1: linear algebra—XY and YX have same eigenvalues, etc.

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Proof 1: linear algebra—XY and YX have same eigenvalues, etc.
Proof 2: Schur complements—

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - xy & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - yx & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & y \\ x & 1 \end{pmatrix}$$

....

....

$$P(x, y) \begin{pmatrix} 1 - xy & 0 \\ 0 & 1 \end{pmatrix} Q(x, y) = \begin{pmatrix} 1 - yx & 0 \\ 0 & 1 \end{pmatrix}$$

(with $\det P(x, y), \det Q(x, y) \neq 0$)

Say f_1, f_2 are **stably associated** if there exist always-invertible matrix polynomials $P(x), Q(x)$ so that

$$P(x)\begin{pmatrix} f_1(x) & 0 \\ 0 & 1_{m_1} \end{pmatrix} Q(x) = \begin{pmatrix} f_2(x) & 0 \\ 0 & 1_{m_2} \end{pmatrix}$$

So:

If f_1, f_2 are stably associated then $\mathscr{Z}(f_1) = \mathscr{Z}(f_2)$.

Say $f \in \mathbb{M}_d(\mathbb{C} < x >)$ is an **atom** if it does NOT factor into non-invertibles $f = gh$

Theorem (Polynomial Singularitätstellensatz)

Let f_1, f_2 be nc matrix polynomials with $f(0) = I$.

1) *If f_1 is an atom, then $\det f_1(\Omega^{(n)})$ is irreducible for large n.*
2) *If f_1, f_2 are atoms and $\mathscr{Z}(f_1) = \mathscr{Z}(f_2)$, then f_1 and f_2 are stably associated.*
3) *If $\mathscr{Z}(f_1) \subset \mathscr{Z}(f_2)$ then each atomic factor of f_1 is stably associated to an atomic factor of f_2.*

The big idea: "linearize" the problem

A **(monic) linear pencil**:

$$L(x_1, \dots, x_g) = I_d - \sum_{j=1}^{g} A_j x_j$$

for some $d \times d$ matrices $A_1, \dots, A_g$.
FACT: every nc polynomial $f(x)$ has a **realization**

$$f(x) = c^t L(x)^{-1} b$$

for some monic pencil $L(x)$ of some size d; some vectors $b, c \in \mathbb{C}^d$

- call it **minimal** if the size d is smallest possible
- minimal realizations are unique up to change of basis
- the A's in the pencil of a minimal realization will be jointly nilpotent

$f(x) = c^t L(x)^{-1} b$

Suppose $f(0) \neq 0$. Consider $f(x)^{-1}$.

FACT 1: [Schützenberger 1963] $f(x)^{-1}$ also has a realization (of some size d'):

$$f(x)^{-1} = \widetilde{c}^t \widetilde{L}(x)^{-1} \widetilde{b}$$

FACT 2: [Volčič 2017] If the realization for $f(x)^{-1}$ is minimal, then $\widetilde{L}(x)$ is invertible if and only if $f(x)$ is invertible.

THUS:

$$\det f(x) = 0 \text{ if and only if } \det \widetilde{L}(x) = 0, \text{ that is....}$$

$$\boxed{\mathscr{Z}(f) = \mathscr{Z}(\widetilde{L})}$$

$$\text{“zeroes” of } f(x) \longleftrightarrow \text{“poles” of } f(x)^{-1}$$

$$\det f(x) = 0 \longleftrightarrow \det L(x) = 0$$

Lemma

If f is an nc polynomial then f is stably equivalent to a monic linear pencil L.

Lemma

If f is an **atom**, *then f is stably equivalent to an* **irreducible** *monic linear pencil.*

irreducible means: the coefficients $I, A_1, \ldots, A_g$ of L generate the full matrix algebra $\mathbb{M}_d$

Theorem (Klep-Volčič 2017)

If L_1, L_2 are **irreducible** *monic linear pencils and $\mathscr{Z}(L_1) = \mathscr{Z}(L_2)$, then L_1 is similar to L_2.*

Theorem (Polynomial Singularitätstellensatz)

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Proof of (2):

- $f_1 \sim L_1, f_2 \sim L_2$, both L_i irreducible
- Since $\mathscr{Z}(L_1) = \mathscr{Z}(L_2)$, by [KV17] we have $L_1 \sim L_2$
- $f_1 \sim L_1 \sim L_2 \sim f_2$ □

Eventual irreducibility:

$$\mathscr{Z}_n(f) = \{X \in \mathbb{M}_n(\mathbb{C})^g : \det f(X) = 0\}$$

Introduce variables for all the matrix entries of X:

$$\Omega^k = (\omega_{ij}^k), \quad k = 1, \dots, g, \quad i, j = 1, \dots, n$$

Thus, at each level n the zero locus $\mathscr{Z}_n$ is the zero variety of the polynomial

$$\det f(\Omega^{(n)})$$

in gn^2 complex variables.

We already know:

$$\mathscr{Z}_n(f) = \mathscr{Z}_n(L)$$

for some monic pencil L, and we can choose L irreducible if f is irreducible.

Theorem (C)

Let $L = I - \sum A_j x_j$ be an irreducible monic pencil. Then there is an integer n_0 so that

$$\det L(\Omega_1^{(n)}, \dots, \Omega_g^{(n)})$$

is an irreducible polynomial for all $n \geq n_0$.

Theorem (Polynomial Singularitätstellensatz)

Let f_1, f_2 be nc matrix polynomials with $f(0) = I$.

1) *If f_1 is an atom, then $\det f_1(\Omega^{(n)})$ is irreducible for large n.*
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"Large n" is necessary:

$f(x, y) = (1 - x)^2 - y^2$ is irreducible as an nc polynomial, but

at level 1:

$$(1 - z)^2 - w^2 = (1 - z - w)(1 - z + w)$$

"flip-poly" pencils:

We already know:

$$\mathscr{Z}_n(f) = \mathscr{Z}_n(L)$$

for some monic pencil L (e.g. L the pencil in a minimal realization of f^{-1})

Which pencils L arise this way?

Say a pencil $L = 1 - \sum A_j x_j$ is **flip-poly** if

$$A_j = N_j + E_j, \quad \text{where}$$

- N_j are nilpotent
- E_j are rank one
- $\operatorname{codim} \cap \ker E_j \leq 1$

Lemma

Let $f \in \mathbb{C}\langle x \rangle$ *with* $f(0) = 1$. *Let* L *be the pencil in a minimal realization of* f^{-1}. *Then*

- L *is flip-poly, and*
- $\det f(\Omega^{(n)}) = \det L(\Omega^{(n)})$.

Theorem

$\mathscr{Z}(L) = \mathscr{Z}(f)$ *for some nc polynomial* f *if and only if* $\mathscr{Z}(L) = \mathscr{Z}(L_0)$ *for some flip-poly pencil* L_0.

One more theorem, which relates invariant subspaces for the pencil L in a minimal realization of f to invariant subspaces of the pencil $\widetilde{L}$ in a minimal realization of f^{-1}.....

Applications:

- factoring nc polynomials:

$$f(x)^{-1} = 1 + \begin{pmatrix} c_1 & c_2 \end{pmatrix}^t \begin{pmatrix} L_1 & \star \\ 0 & L_2 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

- smooth points on free loci
- boundaries of spectrahedra