J.E. Pascoe's Noncommutative Free Universal Monodromy Theorem & Applications

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Analytic Extensions

Let $D, \Omega \subseteq \mathbb{C}$ be domains with D strictly contained in Ω.

Question: If f is analytic on D, under what conditions does f extend to be analytic on Ω?

Analytic Continuation Along Curves

Let $\gamma : [0,1] \to \mathbb{C}$ be a curve and f an analytic function defined on an open disk D containing $\gamma(0)$

An **analytic continuation** of (f, D) along γ is a collection of pairs (f_t, D_t) for $t \in (0,1)$ such that

- $f_0 = f$ and $D_0 = D$
- Each D_t is an open disk centered at $\gamma(t)$ and f_t is analytic on D_t
- For each t, there is an $\epsilon > 0$ such that if $|t - t'| < \epsilon$, $\gamma(t') \in D_t$ and $f_t = f_{t'}$ on $D_t \cap D_{t'}$.

Example

Ex. Let $f(z) = \mathrm{Log}(z) = \log|z| + i\mathrm{Arg}(z)$, near $z = 1$ where $\mathrm{Arg}(z) \in [-\pi, \pi)$

- $\gamma_1(t) = e^{i\pi t}$, for $t \in [0, 1]$
- $f_1(z) := \log|z| + i\arg(z)$ with $\arg(z) \in [-\pi/2, 3\pi/2)$.

- $\gamma_2(t) = e^{-i\pi t}$, for $t \in [0, 1]$
- $f_2(z) := \log|z| + i\arg(z)$ with $\arg(z) \in [-3\pi/2, \pi/2)$

Monodromy Theorem

Two curves $\gamma_0, \gamma_1 : [0,1] \to \Omega$ are **fixed endpoint homotopic** if

$$\gamma_0(0) = a = \gamma_1(0) \text{ and } \gamma_0(1) = b = \gamma_1(1)$$

and if there is a continuous function $\Gamma(t,s) : [0,1] \times [0,1] \to \Omega$ such that

$$\Gamma(t,0) = \gamma_0(t), \quad \Gamma(t,1) = \gamma_1(t), \quad \Gamma(0,s) = a, \quad \Gamma(1,s) = b.$$

Monodromy Theorem #1

Let f be analytic on $D \subseteq \Omega$ and assume that f analytically continues along each curve $\gamma \subseteq \Omega$ that begins in D.

If $\gamma_0, \gamma_1 : [0,1] \to \Omega$ are fixed endpoint homotopic curves starting in D, then the analytic continuations of f along γ_0, γ_1 agree in a neighborhood of $\gamma_j(1)$.

Monodromy Theorem #2

Let Ω be simply connected, let f be analytic on $D \subseteq \Omega$, and assume that f analytically continues along each curve $\gamma \subseteq \Omega$ that begins in D.

Then there is an analytic function $F : \Omega \to \mathbb{C}$ that agrees with f on D.

Simply Connected is Required Here

Ex. Let $f(z) = \mathrm{Log}(z) = \log|z| + iArg(z)$, in a neighborhood of $z = 1$ where $Arg(z) \in [-\pi, \pi)$

- f analytically continues along each curve in $\mathbb{C} \setminus \{0\}$.
- f does not extend to a globally analytic function F on $\mathbb{C} \setminus \{0\}$.
- The analytic continuations of f along different curves do not have to agree.

Free Sets

For fixed $d \in \mathbb{N}$, the **matrix universe** $\mathcal{M}^d$ (for this talk) is the collection of all d-tuples of matrices of the same size:

$$\mathcal{M}^d := \bigcup_{n=1}^{\infty} M_n(\mathbb{C})^d.$$

A **free set** $D \subseteq \mathcal{M}^d$ is set that satisfies the following

- $X, Y \in D$ implies $X \oplus Y \in D$
- $X \in D$ and V unitary implies $VXV^* \in D$.

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Examples:

- $D = \mathcal{M}^d$
- $\mathbb{A} = \{X \in \mathcal{M}^1 : \|X\|, \|X^{-1}\| < 2\}$
- Free sets can be built from polynomial inequalities:

$$S = \{(X_1, X_2) \in \mathcal{M}^2 : \|X_1^2 + 2X_2X_1 - X_2X_1\| < 1\}$$

A free set $D \subseteq \mathcal{M}^d$ is **a noncommutative domain** if for each n, $D_n := D \cap M_n(\mathbb{C})^d$ is both open and connected.

Free Functions

$f : D \to \mathcal{M}^{\tilde{d}}$ is a **free function** if f

- $X \in D_n$ implies $f(X) \in M_n(\mathbb{C})^{\tilde{d}}$.
- If $X, Y \in D$, then $f(X \oplus Y) = f(X) \oplus f(Y)$.
- If $X, S^{-1}XS \in D$, then $f(S^{-1}XS) = S^{-1}f(X)S$.

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Examples

- Non-commutative polynomials $p \in \mathbb{C}[X_1, \dots, X_d]$,
$$p(X_1, X_2, X_3) = X_1X_2 - X_2X_1 + 2X_2X_3X_1.$$
- Noncommutative rational functions,
$$f(X_1, X_2) = \left(I - X_2(X_1^2 - X_2X_1)^{-1}\right)^{-1}.$$

Analyticity

- A free function f is analytic if each $f_n := f|_{D_n}$ is analytic.
- f is analytic if each f_n is continuous (if each f_n is locally bounded), e.g. Helton-Klep-Mccullough, 2011.

Monodromy Theorem

Let D, Ω be noncommutative domains in $\mathcal{M}^d$ with $D \subsetneq \Omega$.

γ is a curve in an NC domain Ω if γ is a standard curve in $\Omega_n := \Omega|_{\Omega \cap M_n(\mathbb{C})^d}$ for some n, i.e. $\gamma : [0,1] \to \Omega_n$ for some n.

Main Theorem (Pascoe 2020)

Let f be a free analytic function on D and assume that f analytically continues along each curve $\gamma \subset \Omega$ that begins in D.

Then there is a free analytic function F on Ω that agrees with f on D.

Critical Point: Ω does not need to be simply connected!

Three Proofs

- "Free" Proof
- "Disk Bounding" Proof
- "Sphere Embedding" Proof

Preliminaries

Main Theorem (Pascoe 2020)

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Preliminary Observations

Let γ_1, γ_2 be curves in Ω_n starting in D_n and $\hat{\gamma} = \begin{bmatrix} \gamma_1 & \\ & \gamma_2 \end{bmatrix}$ in Ω_{2n}.

Let F_1, F_2, $\hat{F}$ denote the analytic continuations of f along the curves $\gamma_1, \gamma_2, \hat{\gamma}$.

- $F_1\big(\gamma_1(t)\big) \in M_n(\mathbb{C})^{\tilde{d}}$
- $\hat{F}(\hat{\gamma}(t)) = \begin{bmatrix} F_1(\gamma_1(t)) & \\ & F_2(\gamma_2(t)) \end{bmatrix}$
- If $S \in GL_n(\mathbb{C})$, then $F_1\big(S\gamma_1(t)S^{-1}\big) = SF_1\big(\gamma_1(t)\big)S^{-1}$

Basic Proof Idea

Summary. Analytic continuations are graded, respect direct sums, respect similarities.

Vague Key Goal: Show that analytic continuations along different curves agree with each other, so F can be globally defined on Ω.

Set-up for all proofs

Let $\gamma_1, \gamma_2 : [0,1] \to \Omega_n$ satisfy

- $\gamma_1(0) = a = \gamma_2(0) \in D_n$
- $\gamma_1(1) = b = \gamma_2(1) \in \Omega_n$

Let F_1, F_2 be the analytic continuations of f along γ_1, γ_2

Key Goal: Show that $F_1(b) = F_2(b)$.

“Free” Proof

Define the curve: $$\hat{\gamma}(t) = \begin{bmatrix} \gamma_1(t) & \\ & \gamma_2(t) \end{bmatrix}.$$

Let F denote the formula for the analytic continuation of f along $\hat{\gamma}$.

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Let F denote the formula for the analytic continuation of f along $\hat{\gamma}$.

Choose $\epsilon > 0$, so that F also gives an analytic continuation of f along

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) & \epsilon \frac{\gamma_1(t)-\gamma_2(t)}{\|\gamma_1(t)-\gamma_2(t)\|^{1/2}} \\ & \gamma_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\epsilon}{\|\gamma_1(t)-\gamma_2(t)\|^{1/2}} \\ 0 & 1 \end{bmatrix}^{-1} \hat{\gamma}(t) \begin{bmatrix} 1 & \frac{\epsilon}{\|\gamma_1(t)-\gamma_2(t)\|^{1/2}} \\ 0 & 1 \end{bmatrix}.$$

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Then

$$\begin{aligned} F(\gamma(t)) &= \begin{bmatrix} 1 & \frac{\epsilon}{\|\gamma_1(t) - \gamma_2(t)\|^{1/2}} \\ 0 & 1 \end{bmatrix}^{-1} F(\hat{\gamma}(t)) \begin{bmatrix} 1 & \frac{\epsilon}{\|\gamma_1(t) - \gamma_2(t)\|^{1/2}} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{\epsilon}{\|\gamma_1(t) - \gamma_2(t)\|^{1/2}} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} F_1(\gamma_1(t)) & \\ & F_2(\gamma_2(t)) \end{bmatrix} \begin{bmatrix} 1 & \frac{\epsilon}{\|\gamma_1(t) - \gamma_2(t)\|^{1/2}} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} F_1(\gamma_1(t)) & \epsilon \frac{F_1(\gamma_1(t)) - F_2(\gamma_2(t))}{\|\gamma_1(t) - \gamma_2(t)\|^{1/2}} \\ & F_2(\gamma_2(t)) \end{bmatrix}. \end{aligned}$$

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Letting $t \to 1$ shows $F_1(b) = F_2(b)$ since otherwise, $F(\gamma(1))$ is undefined.

"Disk Bounding Proof"

Define curves

$$\hat{\gamma}(t) = \begin{bmatrix} \gamma_1(t) & \\ & \gamma_2(t) \end{bmatrix} \quad \text{and} \quad \gamma(t) = \begin{bmatrix} \gamma_2(t) & \\ & \gamma_1(t) \end{bmatrix}.$$

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Define $\Gamma : [0,1] \times [0,1] \to \Omega_{2n}$ by

$$\Gamma(t,s) = \begin{bmatrix} \cos(s\pi/2) & \sin(s\pi/2) \\ -\sin(s\pi/2) & \cos(s\pi/2) \end{bmatrix} \begin{bmatrix} \gamma_1(t) & \\ & \gamma_2(t) \end{bmatrix} \begin{bmatrix} \cos(s\pi/2) & -\sin(s\pi/2) \\ \sin(s\pi/2) & \cos(s\pi/2) \end{bmatrix}.$$

Then γ, $\hat{\gamma}$ are fixed endpoint homotopic since:

$$\Gamma(t,0) = \hat{\gamma}(t), \quad \Gamma(t,1) = \gamma(t), \quad \Gamma(0,s) = aI, \quad \Gamma(1,s) = bI.$$

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The standard (multivariable) Monodromy Theorem implies: the analytic continuations of f along γ, $\hat{\gamma}$ must agree near $t = 1$, so

$$\begin{bmatrix} F_1(\gamma_1(1)) & \\ & F_2(\gamma_2(1)) \end{bmatrix} = \begin{bmatrix} F_2(\gamma_2(1)) & \\ & F_1(\gamma_1(1)) \end{bmatrix},$$

or equivalently, $F_1(b) = F_2(b)$.

Sphere Embedding

Define $\mathcal{G} \subseteq \Omega_{2n}$ by

$$\mathcal{G} := \left\{ \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} \gamma_1(t) & \\ & \gamma_2(t) \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} : c, d \in \mathbb{R}, c^2 + d^2 = 1, t \in [0, 1] \right\}.$$

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One can show $\mathcal{G} \cong S^2$, which is simply connected. As

$$\hat{\gamma}(t) = \begin{bmatrix} \gamma_1(t) & \\ & \gamma_2(t) \end{bmatrix} \quad \text{and} \quad \gamma(t) = \begin{bmatrix} \gamma_2(t) & \\ & \gamma_1(t) \end{bmatrix}$$

are curves in $\mathcal{G}$, the classical Monodromy Theorem implies that the analytic continuations of f along those curves must agree at the final endpoint and so, $F_1(b) = F_2(b)$.

Pluriharmonic Functions

Commutative Case

If Ω is domain in $\mathbb{C}^d$, then $u : \Omega \to \mathbb{R}$ is **pluriharmonic** if for all $a \in \Omega$, $b \in \mathbb{C}^d$,

$$\Delta_z u(a+bz)\Big|_{z=0} = \left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(a+bz)\right)\Big|_{z=0} = 0.$$

If Ω is simply connected, $u = Re(f)$ for some f analytic on Ω.

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Noncommutative Case

A self-adjoint valued u is **free pluriharmonic** on an NC domain Ω if

- $X \in \Omega_n$ implies $u(X) \in M_n(\mathbb{C})$.
- $X, Y \in \Omega$, implies $u(X \oplus Y) = u(X) \oplus u(Y)$.
- V unitary, $X \in \Omega$ implies $u(V^* X V) = V^* u(X) V$.
- for all $A \in \Omega_n$, $B \in M_n(\mathbb{C})^d$,

$$\Delta_z u(A+Bz)\Big|_{z=0} = \left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(A+Bz)\right)\Big|_{z=0} = 0.$$

Pluriharmonic Conjugates

Corollary, Pascoe 2020

If u is a pluriharmonic free function defined on a noncommutative domain Ω, then there is a free analytic function F on Ω with $u = \mathrm{Re}(F)$.

Pluriharmonic Conjugates

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If u is a pluriharmonic free function defined on a noncommutative domain Ω, then there is a free analytic function F on Ω with $u = \mathrm{Re}(F)$.

Proof Idea.

- Solve the related PDE ($u = \mathrm{Re}(f)$) in a neighborhood of each point in Ω.
- Patch the solutions together using the Monodromy theorem.

Critical Point: Ω does not need to be simply connected!

Question: What other PDEs are important in the non-commutative setting?

Intuition Check

Annulus: $A = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$.

- $u(z) = \log |z|$ is harmonic on A.
- u is not the real part of an analytic function defined globally on A.

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Noncommutative Annulus: $\mathbb{A} = \{Z \in \mathcal{M}^1 : \|Z\|, \|Z^{-1}\| < 2\}$.

- $u(Z) := \log |Z| = \frac{1}{2} \log(Z^* Z)$ is a well defined (real free) function on $\mathbb{A}$.
- u is **not** pluriharmonic:

$$\Delta_z u \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \Big|_{z=0} = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

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Note: $F : \mathbb{A} \to \mathcal{M}^1$ is free analytic iff on $\mathbb{A}$, $F(Z) = \sum_{n=-\infty}^{\infty} c_n Z^n$.

Corollary, Pascoe 2020

The pluriharmonic free functions on $\mathbb{A}$ are of the form $u(Z) = \mathrm{Re}\left(\sum_{n=-\infty}^{\infty} c_n Z^n \right)$.

Plurisubharmonic Functions

If Ω is an NC domain in $\mathcal{M}^d$, a self-adjoint valued free u on Ω is **plurisubharmonic** if

- u is a real free function (satisfies the first 3 properties of a pluriharmonic function)
- for all $A \in \Omega_n$, $B \in M_n(\mathbb{C})^d$,

$$\Delta_z u(A+Bz)\Big|_{z=0} = \left(\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(A+Bz)\right)\Big|_{z=0} \geq 0.$$

- (Green, Helton, and Vinnikov (2011) and Green (2012)): Noncommutative plurisubharmonic polynomials
- (Dym, Klep, Helton, McCullough and Volcic, 2019): Plurisubharmonic free rational functions
- (Pascoe 2020): Realization formulas for general plurisubharmonic functions
  - Pascoe first proved a local version.
  - Then used Monodromy to obtain a global realization.

Open Question

Existence of Logarithms

If $\Omega \subseteq \mathbb{C}$ is simply connected and $f : \Omega \to \mathbb{C} \setminus \{0\}$ is analytic, then there exist an analytic g on Ω with $f = e^g$.

Open Question: If F is a nonsingular free noncommutative function, when does F possess a logarithm?

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Related Example. Set $F(X_1, X_2) = e^{X_1} e^{X_2}$. Then F is always nonsingular, but standard results in Lie Theory show that F does not have a globally defined logarithm.

Why can't monodromy help? There are curves along which the logarithm solution cannot be analytically continued.

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If $\Omega \subseteq \mathbb{C}$ is simply connected and $f : \Omega \to \mathbb{C} \setminus \{0\}$ is analytic, then there exist an analytic g on Ω with $f = e^g$.

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Why can't monodromy help? There are curves along which the logarithm solution cannot be analytically continued.

Takeaway. The noncommutative situation is simpler in some respects but more complicated in others.

Thanks for listening!

Based on:

J.E. Pascoe. Noncommutative Free Universal Monodromy, Pluriharmonic Conjugates, and Plurisubharmonicity. 2020. Available at https://arxiv.org/abs/2002.07801.