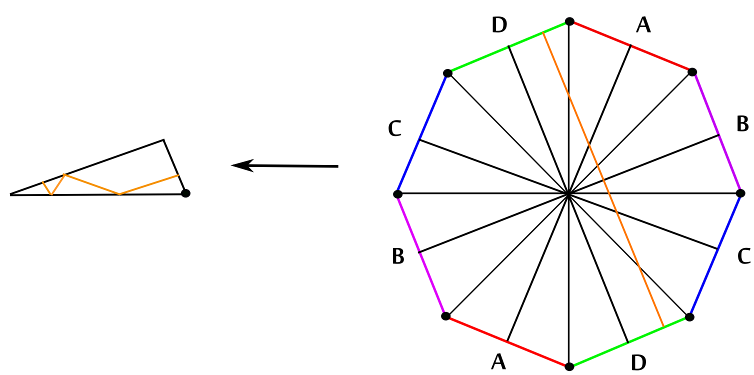
Flat Surfaces and Dynamics of Moduli Space,II

Barak Weiss (Tel-Aviv),
Kasra Rafi (Toronto), Ferran Valdez (Morelia) & Anton Zorich (Paris)

26th May 2016-31th May 2019

1 Overview of the Field

Flat surfaces (also called translation surfaces) are pairs consisting of a Riemann surface together with a holomorphic one-form. The interest in flat surfaces stems from the dynamics on polygonal billiards. Instead of reflecting the trajectory at the boundary of the table one reflects the table and glues them along the reflection edges. The billiard paths become straight lines on the unfolded object and are thus much more easily accesible objects. If all the angles of the table are rational multiples of π, the resulting object is a flat surface. The



general case leads to the study of infinite translation surfaces where much fewer results are currently known.

The set of all flat surfaces can be parametrized by a bundle $\Omega\mathcal{M}_g$ over the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. Neither the moduli space $\mathcal{M}_g$ nor the bundle $\Omega\mathcal{M}_g$ are homogeneous spaces. Nevertheless, the moduli space of flat surfaces $\Omega\mathcal{M}_g$ admits an action of the Lie group $\mathrm{SL}_2(\mathbb{R})$ and this action has many facets that mimic the action of a Lie group on a homogeneous space. On the other hand, the straight line flow on a flat surface is itself a very interesting dynamical system. A lot of the beauty and interest in this topic stems from the interplay interpreting questions about the dynamics of the straight line flow on flat surfaces in terms of the $\mathrm{SL}_2(\mathbb{R})$-action and vice versa. This situation (and its counterpart consisting of quadratic differentials and half-translation surfaces) is pretty unique compared to all known spaces of complex manifolds. However, the field and the conference participants are open to extension to other contexts with similar dynamics interplays.

2 Recent Developments and Open Problems

A major recent development due to Eskin, Mirzakhani and Mohammadi is that the closures of $\mathrm{SL}_2(\mathbb{R})$-orbits are known to be nice manifolds. There are structure theorems that are analogs of Ratner's theorem in homogeneous dynamics (for horocyclic group actions) and that exclude wild orbit closures (as the geodesic flow in homogeneous dynamics may produce). More precisely, the moduli space $\Omega\mathcal{M}_g$ is stratified according to the number and multiplicity of the holomorphic one-form and the $\mathrm{SL}_2(\mathbb{R})$-action respects this stratification. The results of [EsMi] and [EsMiMo] show that such an orbit closure is always cut out by linear equation in the natural coordinate system on strata called period coordinates. Moreover, it was shown in [Fi] that they are algebraic (i.e. quasi-projective) manifolds. Consequently, the *orbit closure problem*, i.e. determining the possible orbit closures for given genus, construction of such orbit closures or finiteness results, is one of the important and currently very active topics in the field. Surfaces with a closed $\mathrm{SL}_2(\mathbb{R})$-orbit are called *Veech surfaces*. Primitive examples of Veech surfaces (such that do not arise from covering constructions from lower genus) are very rare and interesting. More recently, Eskin, McMullen, Mukamel and Wright have found examples of primitive, irreducible $\mathrm{SL}_2(\mathbb{R})$-invariant submanifolds of dimension 4 [MMW, EMMW].

The one-dimensional dynamical systems known as *Interval Exchange Transformations* have been a focus of study for ergodic theorists and geometers for at least the last 40 years, and for the last 20 years the primary methods of study have been to use renormalization dynamics (induction procedures due to Rauzy-Veech, Zorich, Yoccoz, and others) and their connections to dynamics of the $\mathrm{SL}_2(\mathbb{R})$-action on the moduli space of translation surfaces. For example, properties like unique ergodicity [**?**, **?**] and weak mixing [AvFo, AvDe] have been studied using these methods. More recently, Interval Exchange Transformations on a infinite number of intervals, which naturally appear when considering infinite-type flat surfaces, have been studied as well (see e.g. [H]).

The orbit closure problems described above have important applications to counting problems for billiards and translations surfaces, in particular by studying ratios of intrinsic volumes of these orbit closures. This is further connected to the problem of counting square-tiled surfaces and pillowcase covers. Important recent contributions include the works of Eskin–Kontsevich–Möller–Zorich [EsKoZo], [EsKoMoZo] which express the sum of Lyapunov exponents of the Hodge bundle in terms of the area Siegel–Veech constants and establish bounds for individual exponents in terms of the normalized degrees of the holomorphic subbundles. Very recently A. Aggarwal and Chen–Möller–Sauvaget–Zagier obtained spectacular results on large genus asymptotics of Masur–Veech volumes of strata in the moduli spaces of Abelian differentials and on asymptotic values of Siegel–Veech constants proving, in particular, old conjectures of Eskin and Zorich, ([Ag1], [Ag2], [ChMoZa], [ChMoSZa]). The technology of geometry and dynamics in the moduli spaces was successfully applied to long-standing problems of mathematical models in statistical physics and in solid-state physics as in computation of the diffusion rate in the Ehrenfest wind-tree model ([DeHuLe], [DeZo], [Fo]) and in Novikov's problem on electron transport in the inverse lattice in constant magnetic field [AvHuSk].

Open problems. On Tuesday, May 28th afternoon we had a problem session whose outcome is presented in the next paragraphs. We refer readers without the necessary background to the surveys [**?**, FM, **?**].

1. **[**Luca Marchese]. Let π be a (finite) irreducible permutation of a set of d symbols and Δ_π the space of admissible lengths for IETs on d intervals. Given a curve $\gamma : I \to \Delta_\pi$ determine how many points in its image define uniquely ergodic IETs. The ensuing discussion placed this problem as part of the following

   Conjecture [MW]: if $\gamma : I \to \Delta_\pi$ is analytic and not contained in a proper affine subspace then for almost every s the IETS defined by $\gamma(s)$ is uniquely ergodic.

   In [MW] the conjecture is confirmed in the special case that $\gamma(s) = (s, s^2, \ldots, s^d)$ and the permutation π is given by:

$$\begin{matrix} 1 & 2 & \ldots & d \\ d & d-1 & \ldots & 1 \end{matrix}$$

   Rene Rühr informed us that the conjecture is also valid with the same curve, and for any irreducible permutation on d symbols. The conjecture has recently been confirmed in several other cases by Fraczek [Fr] in connection with some questions in mathematical physics.

2. [Anja Randecker] Let M be the Chamanara surface (see [Cham] for details). Is there a direction θ such that:
$$\lim_{t\to\infty} \operatorname{diam}(g_t r_\theta \cdot M) = 0?$$
Here $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ and the action is the standard $\mathrm{GL}(2,\mathbb{R})$-action on translation surfaces. Find a non-periodic direction such that the above limit is different from zero. Are there directions (different from the vertical or horizontal) on which the translation flow on M is uniquely ergodic?

3. [Paul Apisa] Classification of Wollmilchsau-like Veech surfaces.

   Definition: Say that a Veech surface is Q-like if (1) in any cylinder direction there are exactly two cylinders and (2) these cylinders have identical circumferences and heights (i.e. their interiors are congruent).

   Examples: The Wollmilchsau and Ornithorynque are examples. More generally, Matheus-Yoccoz construct infinitely many examples in arbitrarily large genus in Section 3 of [MY]. While Matheus-Yoccoz do not record the fact that these examples are Q-like, it follows from their description of the Veech group (whose upper-half plane quotient has two cusps corresponding to the cylinder decompositions in horizontal and vertical directions, both of which have properties (1) and (2) of the definition above).

   General Facts:

   - Fact 1: By Alex Wright's cylinder deformation theorem, any Q-like Veech surface is square-tiled.
   - Fact 2: Classifying Q-like Veech surfaces can be reduced to the following problem: find all square-tiled surfaces such that (1) the quotient by the absolute period lattice is a degree d map to a flat torus E branched over a subset of the four 2-torsion points P and (2) the preimage of any cylinder on the torus with marked points (E, P) is a single cylinder.

   This second formulation has a purely combinatorial/representation-theoretic formulation. A degree d branched cover of (E, P) can be specified by a representation from $\pi_1(E \setminus P)$ into $\mathrm{Sym}(d)$ - the symmetric group on d letters. The fundamental group of $E \setminus P$ is the free group F_5 on 5 symbols; fix an isomorphism between $\pi_1(E \setminus P)$ and F_5. We are looking for representations from F_5 into $\mathrm{Sym}(d)$ so that all words in F_5 corresponding to core curves of cylinders map to d-cycles.

   Question: Classify all Q-like Veech surfaces. The Matheus-Yoccoz examples are cyclic covers of tori and contain the congruence subbroup $\Gamma(2)$ in their Veech group. Do these properties hold for all Q-like Veech surfaces?

4. [Jon Chaika, Jayadev Athreya] Let u be a saddle connection and v_u its holonomy vector. We say that u has an *ε-friend* w if $||u - w|| < \varepsilon$. Is there an $\varepsilon > 0$ such that for any translation surface (X, ω) and any $\delta > 0$ we have that
$$\limsup_{L\to\infty} \frac{\#\{\text{saddle connections of length} \le L \text{ with an } \varepsilon\text{-friend}\}}{L^2} < \delta?$$

5. [Howard Masur] Let $\mathcal{H}(\hat{m})$ be a connected component of a stratum of abelian differentials in $\Omega\mathcal{M}_g$ and $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}$.

   Question: Classify the u_t-ergodic invariant measures for the action of u_t on $\mathcal{H}(2)$. *Remark*: Alex Eskin is offering a case of wine for the solution.

   Motivation: Furstenberg [Fu] showed that the Haar measure is the only u-invariant measure for the action of u on $\mathrm{SL}(2,\mathbb{R})/\Gamma$, where Γ is a discrete cocompact subgroup.

6. [Phil Engel] An *integral affine surface* S is a surface with charts to $\mathbb{R}^2$ whose transition functions lie in $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$. There is a standard conical singularity of positive curvature determined by the matrix

$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. The following claim (without proof) is due to Kontsevich and Soibelman: the moduli space of $(\mathbb{S}^2, P)$, where P is formed by 24 punctures with standard singularity structure is Hausdorff when considering the topology that makes the period map continuous.

Questions: can this claim be disproved? Is there a straight line triangulation on any surface in the aforementioned moduli space?

7. **Kasra Rafi**: what does the generic translation surface look like?

   E.g. what are the asymptotics as $g \to \infty$ of: (1) the inyectivity radius, (2) Cheeger constants, (3) the diameter of the surface or (4) its covering radius?

   For hyperbolic surfaces much is known thanks to the work of M. Mirzakhani.

   Question: can you get asymptotics as $g \to \infty$ for volumes of strata of quadratic differentials or limiting distributions of Lyapunov exponents?

   Question:

   Given $0 < a < b$ find $\lim_{g\to\infty} \#\{$cylinders of circumference lying in $[\frac{1}{\sqrt{g}}, \frac{b}{\sqrt{g}}]\}$

3 Presentation Highlights

We had 16 research one-hour talks and 2 posters. The vast majority of the participants and speakers were in the first stages of their careers. Attached is the abstract and/or brief synopsis of each lecture, grouped by topic.

3.1 Interval exchange transformations

Jon Chaika talked about joinings defined by IETs. Joinings were introduced by Furstenberg: given two transformations (X, μ, T) and (Y, ν, S), they are are $T \times S$-invariant measures on $X \times Y$ that project on the coordinates to μ and ν respectively. They describe the ways that these transformations 'talk' to each other. In particular, a common factor of these transformations gives rise to a joining between them. His talk was a survey on known results on joinings between different interval exchange transformations and flows on translation surfaces, different powers of the same transformations and self-joinings.

Pat Hooper spoked about infinite rational IETs. Recall that an IET is rational if it translates points by rational amount. In a finite rational IET, all orbits are periodic. In an infinite IETs, this need not be the case. In the examples he discussed it can be proved that almost every point is periodic and that the dynamics on the aperiodic set can be completely understood. Furthermore, as Hooper discussed, the class of infinite IETs in his talk admits a renormalization scheme (though it remains to be seen how effective this scheme is for understanding generic IETs). He covered recent work in colaboarion with Kasra Rafi and Anja Randecker [H1], and a work in progress joint with Anna Tao.

3.2 Affine submanifolds of moduli space

If Q is a connected component of a stratum of quadratic differentials let $\widetilde{Q}$ denote the set of translation surfaces (X, ω) which arise as orientation (double) coverings of elements in Q. This set is an affine invariant submanifold (abbreviated AIS). *Paul Apisa* explained that if $\mathcal{M}$ in the moduli space of translation surfaces of genus $g \geq 2$ and its rank is greater of equal to $\frac{g}{2} + 1$ then $\mathcal{M}$ is $\widetilde{Q}$ for some Q or the whole stratum. He conjectured that this result should hold for AIS whose rank is strictly bigger than $\frac{g}{2}$. The results are joint work with Alex Wright.

Florent Ygouf. Completed the table of affine invariant submanifolds for $\mathcal{H}^{odd}(2, 2)$ and $\mathcal{H}(3, 1)$. The new ingredient is a complete classification in these strata of the rank-one (in the sense of A. Wright) loci. They are all associated with Prym involutions. A major ingredient of Ygouf's analysis is the study of closures of Rel leaves in these genus 3 strata.

3.3 Counting problems and Volumes

Rene Rühr. A central problem is to determine if every translation surface satisfies quadratic growth asymptotics, that is whether the number of saddle connections of length at most L satisfies a bound $cL^2 + o(L^2)$ where $c > 0$ depends on the surface (and is called the *quadratic growth constant*). Three central results are that Veech surfaces satisfy quadratic asymptotics (Veech 1989 [Ve1]), almost every surface satisfies quadratic asymptotics (Eskin-Masur 1991 [EM]), and every surface satisfies quadratic asymptotics if one allows an additional Cesaro average (Eskin-Mirzakhani-Mohammadi 2015 [EsMiMo]). The first two of these results were strengthened for an estimate of the form $cL^2 + O(L^{2-\eta})$ for some $\eta > 0$; for almost every surface this is a result of Nevo, Rühr and Weiss, for Veech surfaces this is a result of Burrin, Nevo, Rühr and Weiss. More refined estimates involving counting saddle connections with $\mod p$ restrictions were also given. This is recent joint work of Magee and Rühr with Gutiérrez-Romo.

Effective estimates of saddle connection counting have a major role in the recent breakthroughs of *Angel Pardo* for counting cylinders in certain infinite translation surfaces associated with the wind-tree model. Recapitulating from the abstract to his talk: recall that the Gauss circle problem consists in counting the number of integer points of bounded length in the plane. This problem is equivalent to counting the number of closed geodesics of bounded length on a flat two dimensional torus. Many counting problems in dynamical systems have been inspired by this problem. For 30 years, the experts try to understand the asymptotic behavior of closed geodesics in translation surfaces and periodic trajectories on rational billiards. (Polygonal billiards yield translation surfaces naturally through an unfolding procedure.) H. Masur proved that this number has quadratic growth rate. We will study the counting problem on infinite periodic rational billiards and translation surfaces. The first example and motivation is the wind-tree model, a $\mathbb{Z}^2$-periodic billiard model. In the classical setting, we place identical rectangular obstacles in the plane at each integer point and we play billiard on the complement. It is possible to give quite precise results on the counting problem for this model, thanks to the many symmetries it presents. These results, however, do not extend to more general contexts. We present a general result on the counting problem for infinite periodic translation surfaces that uses new ideas: a dynamical analogous, for the algebraic hull of a cocycle, to strong and super-strong approximation on algebraic groups. Under these approximation hypothesis we exhibit asymptotic formulas for the number of closed geodesics of bounded length on infinite periodic translation surfaces. Finally, we present some work in progress: applications of the aforementioned results and discuss why we think these hypothesis hold in general.

Elise Goujard presented a formula for the Masur-Veech volumes of the principal strata of quadratic differentials (as well as Siegel-Veech constants) in terms of intersection numbers involving ψ-classes. Her formula closely relates the counting of square-tiled surfaces with Mirzakhani's counting of simple closed geodesic multicurves on hyperbolic surfaces, and leads to several conjectures for the large genus asymptotics. Her results are joint work with V. Delecroix, P. Zograf and A. Zorich.

Francisco Arana-Herrera revisited and improved results of Maryam Mirzakhani's thesis. Namely, let X be a closed, connected, hyperbolic surface of genus 2. Is it more likely for a simple closed geodesic on X to be separating or non-separating? How much more likely? In her thesis, Mirzakhani gave very precise answers to these questions. One can ask analogous questions for square-tiled surfaces of genus 2 with one horizontal cylinder. Is it more likely for such a square-tiled surface to have separating or non-separating horizontal core curve? How much more likely? Recently, Delecroix, Goujard, Zograf, and Zorich gave very precise answers to these questions. Surprisingly enough, their answers were exactly the same as the ones in Mirzakhanis work. In this talk we explore the connections between these counting problems, showing they are related by more than just an accidental coincidence.

3.4 Large genus asymptotics

Amol Aggarwal. Let $\mathfrak{C} \subset \mathcal{H}(\hat{m})$ be a connected component of a strata of abelian differentials and $c_{area}(\mathfrak{C})$ be the corresponding area Siegel-Veech constant (see [Ag2] and references there is for details). The main result in this talk is the following formula:

$$\frac{1}{2} - \frac{c}{g} < c_{area}(\mathfrak{C}) < \frac{1}{2} + \frac{c}{g}$$

for some constant $c > 0$. This formula was predicted by Eskin and Zorich in 2003.

3.5 Global properties of Strata

Martin Möller's talk presented recent progress on the computation of the Euler characteristic of strata of abelian differentials. He started by recalling how to compute the (orbifold) Euler characteristic of a compact orbifold B with a normal crossing divisor D using the top-Chen class of the cotangent bundle Ω^1_B or of the logarithmic cotangent bundle $\Omega^1_B(\log D)$. The main results that he presented where a recursive formula to compute the chern character of the logarithmic cotangent bundle (work in collaboration with Costantini and Zachhuber) and a precise formula for the top Chern class of this bundle in dimension 2.

Aaron Calderon began recalling Kontsevich and Zorich's famous classification of the connected components of strata of translation surfaces over moduli space. The corresponding problem for marked surfaces requires the analysis of which mapping classes can be realized as deformations lying inside the stratum. Calderon presented joint work with Nick Salter in which they classify the (non-hyperelliptic) connected components of strata of marked surfaces (that is, strata considered as bundles of holomorphic one-forms over Teichmüller space). Unlike the case for strata over moduli space, they find that there can be many (but finitely many) connected components, depending on both genus and cone angle.

Benjamin Dosier's talk started with the following definition: a subset $S \subset \mathcal{M}_g$ is *coarsely dense* with respect to the Teichmüller metric if there exist a $K > 0$ such that the K-neighbourhood $N_K(S)$ around S is equal to $\mathcal{M}_g$. For any strata of abelian differentials $\mathcal{H}(\hat{k})$ on surfaces of genus $g > 0$ there is a natural forgetful map $\pi : \mathcal{H}(\hat{k}) \to \mathcal{M}_g$. In analytic terms $\pi(X, \omega) = X$. The main result of this talk (in collaboration with E. Sapir) is that $\pi(\mathcal{H}(\hat{k}))$ is coarsely dense if and only if $dim_{\mathbb{C}}\mathbb{P}\mathcal{H}(\hat{k}) \geq 3g - 3$. The proof uses a compactification of strata due to Bainbridge-Chen-Gendron-Grushevsky-Möller.

3.6 q-differentials

Marissa Loving addressed the follwing general question: when geometric structures on surfaces are determined by the lengths of curves, it is natural to ask which curves lengths do we really need to know? It is a classical result of Fricke that a hyperbolic metric on a surface is determined by its marked simple length spectrum. More recently, Duchin, Leininger and Rafi proved that a flat metric induced by a unit-norm quadratic differential is also determined by its marked simple length spectrum. Loving described a generalization of the notion of simple curves to that of q-simple curves, for any positive integer q, and showed that the lengths of q-simple curves suffice to determine a non-positively curved Euclidean cone metric induced by a q-differential metric.

Philip Engel started by recalling that a group acting on an elliptic curve must have order N = 1, 2, 3, 4, or 6. The corresponding quotient is called an *elliptic orbifold*. Certain branched covers of the order N elliptic orbifold are in bijection with tiled surfaces, and form a lattice in the moduli space of N-ic differentials on Riemann surfaces. The enumerative theory of these branched covers suggests a phantom "elliptic orbifold" for all integers N. I Engel discussed work in progress with Peter Smillie proposing a definition for the Hurwitz theory of this non-existent object, and attempts to relate it to quasi-crystals in the moduli space of quintic differentials and the enumeration of Penrose-tiled Riemann surfaces.

3.7 Polygonal billiards

Christopher Leininger explained joint work with Moon Duchin, Viveka Erlandsson, and Chandrika Sadanand: how the shape of a Euclidean polygon is essentially determined by the symbolic coding of the "generalized diagonal" billiard trajectories. The bulk of the talk involved the reduction of this theorem to a rigidity result for flat surfaces.

4 Scientific Progress and Meeting Outcomes

This was the second edition of a conference on Flat Surfaces and Dynamics of Moduli Space, two years after M. Mirzakhani passed away. The influence of her mathematics was highlighted in several talks in the conference, specially those dealing with counting problems and volumes. A vast majority of the participants were researches in early stages of their career. Moreover, 5 graduate students gave talks. On the other hand, 4 participants came from institutions in Mexico (only one from Mexico City). We believe these facts had a direct impact on the resilience of the mathematical community related to Flat Surfaces in Mexico and abroad. The meeting facilitated interesting collaborations and investigations into new phenomena, as well as illuminating new and unexpected connections. It also helped several groups of researchers to continue working in their current projects together.

References

[Ag1] A. Aggarwal, Large Genus Asymptotics for Volumes of Strata of Abelian Differentials, `arXiv:1804.05431`.

[Ag2] A. Aggarwal, Large Genus Asymptotics for Siegel–Veech Constants, to appear in *GAFA*, `arXiv:1905.10352`.

[AvDe] A. Avila, V. Delecroix: *Weak mixing directions in non-arithmetic Veech surfaces*, J. Amer. Math. Soc. **29** (2016), no. 4, 1167–1208.

[AvHuSk] A. Avila, P. Hubert, A. Skripchenko, Diffusion for chaotic plane sections of 3-periodic surfaces, *Invent. Math.* **206** (2016), no. 1, 109–146.

[AvFo] A. Avila, G. Forni: *Weak mixing for interval exchange transformations and translation flows* , Annals of Math. **165** (2007), 637–664.

[Cham] R. Chamanara: *Affine automorphism groups of surfaces of infinite type*, In the tradition of Ahlfors and Bers, III. Proceedings of the 2nd Ahlfors-Bers colloquium, Storrs, CT, USA, October 18–21, 2001 Contemp. Math., 355, Amer. Math. Soc., Providence, RI, 2004.

[ChMoZa] D. Chen, M. Möller, and D. Zagier. Siegel-Veech constants and quasimodular forms, *J. Amer. Math. Soc.* **31** (2018), no. 4, 1059–1163.

[ChMoSZa] D. Chen, M. Möller, A. Sauvaget, D. Zagier, *Masur–Veech volumes and intersection theory on moduli spaces of abelian differentials*; `arXiv:1901.01785`.

[DeHuLe] V. Delecroix, P. Hubert, S. Lelièvre, Diffusion for the periodic wind-tree model, *Annales scientifiques de l'ENS* **47:6** (2014), 1085–1110.

[DeZo] V. Delecroix, A. Zorich, Cries and whispers in wind-tree forests, to appear in *Proceedings of the conference "What's Next? The mathematical legacy of Bill Thurston*, Princeton University Press, 2020; `arXiv:1502.06405`.

[EsKoZo] A. Eskin, M. Kontsevich, A. Zorich, Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow., *Publ. mathématiques de l'IHÉS* **120**, (2014), pp 207–333.

[EsKoMoZo] A. Eskin, M. Kontsevich, M. Möller, A. Zorich, Lower bounds for Lyapunov exponents of flat bundles on curves, *Geometry and Topology*, **22:4** (2018), 2299–2338.

[EM] A. Eskin and H. Masur, *Asymptotic formulas on flat surfaces*, Ergodic Theory Dynam. Systems, **21** (2001) 443–478.

[EMMW] A. Eskin, C. McMullen, R. Mukamel and A. Wright, *Billiards, quadrilaterals and moduli spaces*, preprint (2019).

[EsMi] A. Eskin, M. Mirzakhani: *Invariant and stationary measures for the* $\mathrm{SL}(2,\mathbb{R})$ *action on moduli space.*, preprint, arXiv:1302.3320, (2013)

[EsMiMo] A. Eskin, M. Mirzakhani, A. Mohammadi: *Isolation, Equidistribution, and Orbit Closures for the* $\mathrm{SL}(2,\mathbb{R})$ *action on moduli space*, preprint, arXiv:1305.3015, to appear in Annals of Math., (2013)

[Fi] S. Filip: *Splitting mixed Hodge structures over affine invariant manifolds*, Annals of Math. **183** (2016), 681–713.

[FM] G. Forni and C. Matheus, *Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards,* J. Mod. Dyn. (2014) **8** 271–436.

[Fo] Ch. Fougeron, Diffusion rate of windtree models and Lyapunov exponents, to appear in Bulletin de la SMF, `arXiv:1803.10717`.

[Fr] K. Fraczek, *Recurrence for smooth curves in the moduli space and application to the billiard flow on nibbled ellipses*, preprint (2019) `https://arxiv.org/abs/1904.12715`

[Fu] H. Furstenberg, *The unique ergodigity of the horocycle flow*, in Recent Advances in Topological Dynamics, Proceedings of the Conference on Topological Dynamics, held at Yale University, June 1923, 1972, in honor of Professor Gustav Arnold Hedlund on the occasion of his retirement, Springer Lecture Notes in Mathematics **318**.

[H] W. P. Hooper, *The Invariant Measures of some Infinite Interval Exchange Maps*, Geometry and Topology **19** (2015) 1895–2038.

[H1] W. P. Hooper, *Renormalizing an infinite rational IET*. preprint (2019) `https://arxiv.org/abs/1808.09989`

[MY] C. Matheus Santos and J-Ch. Yoccoz, *The action of the affine diffeomorphisms on the relative homology group of certain exceptionally symmetric origamis,* Journal of Modern Dynamics, vol. **4**, n. 3, 453–486 (2010).

[MMW] C. McMullen, R. Mukamel and A. Wright, *Cubic curves and totally geodesic subvarieties of moduli space* , Ann. Math., to appear.

[MW] Y. Minsky and B. Weiss, *Cohomology classes represented by measured foliations, and Mahler's question for interval exchanges*, Annales Sci. de L'ENS **2** (2014) 245–284.

[Ve1] W. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. **97** (1989), no. 3, 553–583.