# Ion-Laser Interactions and the Rabi Model

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# ED JAYNES' STEAK DINNER PROBLEM II

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From Physics & Probability, Grandy Jr. & Milonni, eds.

ABSTRACT. During the Spring of 1966, Ed Jaynes presented a seminar course on quantum electronics that included the now famous "Jaynes-Cummings Model" and his Neoclassical Theory (NCT). As part of this seminar series, the NCT description of a two-level atom in an applied field was formulated as a formidable set of coupled nonlinear differential equations. Undaunted, Ed posted the equations on the Washington University Physics Department bulletin board and offered a prize of "a steak dinner for two" at a restaurant of the choice of the person who solves the equations. Within days, Bill Mitchell was able to present an elegant solution at one of the quantum electronics seminars. This early success of a new approach to doing theoretical physics encouraged Ed to challenge the knowledge hungry Physics Department with Steak Dinner Problem II. This problem was a specific mathematical formulation of the exact (i.e. without the Rotating Wave Approximation) description of the interaction of a two-level atom with a single quantized electromagnetic field mode. Jaynes' formulation of the problem appears to have anticipated the use of Bargmann Hilbert space in QED. This problem has remained unsolved for 26 years in spite of the efforts of numerous researchers, most of whom were probably unaware of Jaynes' offered prize. Recent efforts to solve this problem will be described.

model[17] for a sodium chloride lattice. For this application, Born found that the frequency spectrum had two branches, separated by a "no pass" zone where there were no eigenfrequencies. The lower branch is called the acoustical branch and corresponds to motion of particles such that in each short section of the line all particles move in the same direction at any instant. The upper branch is called the optical branch and corresponds to one or more types of particles moving in the direction opposite to that of the others at any given instant.

The analogy between the linear diatomic lattice and the QED two-level problem is not complete because the QED interaction matrix elements increase as $\sqrt{n}$ in contrast to the fixed mechanical spring constants in the corresponding lattice problem. In addition to this, the QED problem has diagonal matrix elements that are the sum of a term that increases as n and a term that alternates between the two unperturbed energy eigenvalues. In contrast the diagonal matrix elements of the diatomic lattice problem just alternate between two values. Graham and Höhnerbach[11] predict a band structure of the eigenvalue spectrum,

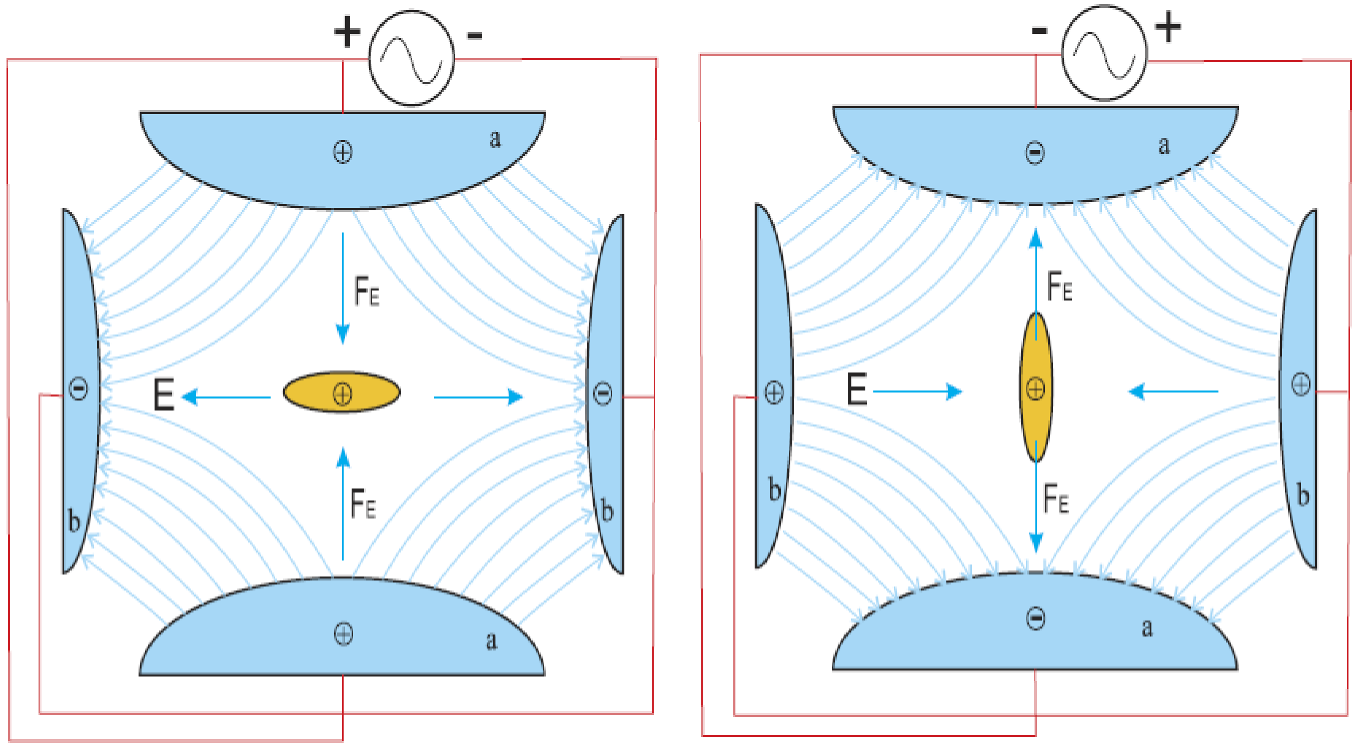



**Fig. 2.** Scheme of a Paul trap to store charged particles using oscillating electric fields generated by a quadrupole. The figure shows two states during an alternate current cycle.

$$\Phi = \frac{U_0 + V_0 \cos \Omega t}{r_0^2 + 2z_0^2}(r^2 - 2z^2),$$

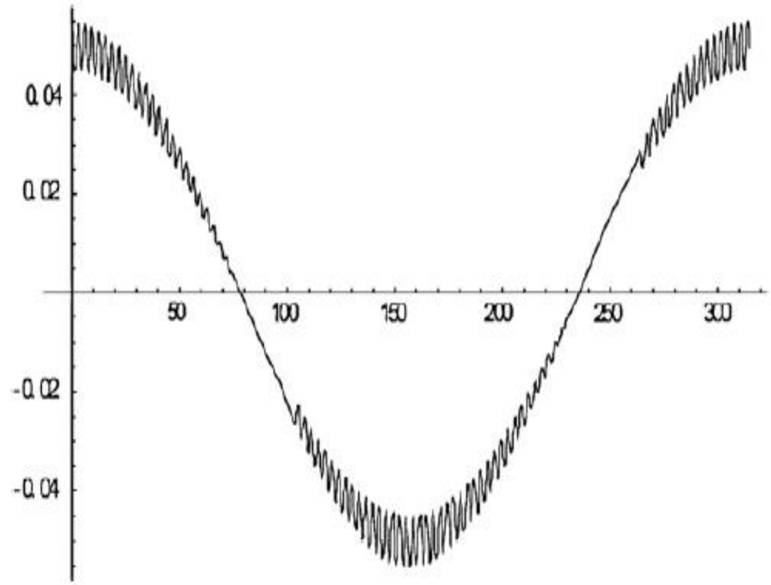



**Fig. 4.** Micro motion and secular motion of a trapped ion with parameters $q = 0.2$, $\beta = 0.02$. The oscillations at high frequency are the micro motion and those at low frequency are the secular motion.

Time dependent ion laser Hamiltonian

$$H = \frac{1}{2}\left[p^2 + \nu^2(t)x^2\right] + \frac{1}{2}\omega_{21}\sigma_z + \lambda\left[E^{(-)(x,t)}\sigma_- + \mathrm{H.C.}\right]$$

$$i\frac{\partial}{\partial t}|\xi(t)\rangle = H|\xi(t)\rangle. \qquad |\phi(t)\rangle = T_{\mathrm{SD}}(t)|\xi(t)\rangle,$$

Classical time dependent HO

$$\ddot{q}+\Omega^2(t)q=0$$

Ermakov-Lewis invariant

Lewis, PRL (1967).

$$I=\frac{1}{2}\left[\left(\frac{q}{\rho}\right)^2+(\rho\dot{q}-q\dot{\rho})^2\right],$$

Ermakov equation

$$\ddot{\rho}+\Omega^2(t)\rho=1/\rho^3$$

Translation to quantum

$$\hat{I} = \frac{1}{2}\left[\left(\frac{\hat{q}}{\rho}\right)^2 + \left(\rho\hat{p} - \dot{\rho}\hat{q}\right)^2\right]$$

Squeezing & Displacement

$$H = \frac{1}{2}\left(\hat{p}^2 + \Omega^2(t)\hat{q}^2\right)$$

$$\frac{\mathrm{d}\hat{G}}{\mathrm{d}t} = \frac{\partial\hat{G}}{\partial t} - \frac{\mathrm{i}}{\hbar}[\hat{G}, \hat{H}(t)]$$

$$\hat{S} = e^{i\frac{\ln\rho}{2}(\hat{q}\hat{p}+\hat{p}\hat{q})} \qquad \hat{D} = e^{-i\frac{\dot{\rho}}{2\rho}\hat{q}^2}$$

H. Moya-Cessa and M. Fernández Guasti
PHYSICS LETTERS A 311, 1 (2003).

$$|\psi> = \hat{S}\hat{D}|\varphi> \equiv \hat{T}|\varphi>$$

$$i\frac{\partial|\psi>}{\partial t} = H|\psi>$$

$$|\psi> = \hat{S}\hat{D}|\varphi> \equiv \hat{T}|\varphi>$$

$$i\frac{\partial|\varphi>}{\partial t} = H_0|\varphi>, \qquad H_0 = \frac{1}{\rho^2(t)}\frac{\hat{p}^2+\hat{q}^2}{2} \equiv \nu(t)\left(a^\dagger a + \frac{1}{2}\right)$$

$$a = \frac{\hat{q}+i\hat{p}}{\sqrt{2}}$$

Time dependence now as a factor

$$|\psi(t)> = \hat{T}\exp\left\{-i\int dt\,\nu(t)\left(a^\dagger a + \frac{1}{2}\right)\right\}|\psi(0)>$$

$$H = \nu\hat{n} + \frac{\omega_{21}}{2}\sigma_z + \lambda E_0\left[e^{i(kx-\omega t)}\sigma_+ + e^{-i(kx-\omega t)}\sigma_-\right]$$

$$\nu\hat{n} \rightarrow \frac{\hat{p}^2}{2} + \nu^2(t)\frac{\hat{x}^2}{2}$$

Via the transformation T, one can cast the above Hamiltonian into a Rabi Hamiltonian with time dependent parameters.

$$H_\omega = \frac{1}{2\nu_0\rho^2(t)}(p^2 + \nu_0^2 x^2) + \frac{1}{2}(\omega_{21} - \omega)\sigma_z + \Omega(t)\left\{\exp\left[-i(\hat{a} + \hat{a}^\dagger)\eta(t)\right]\sigma_- + \text{H.C.}\right\}$$

# Optical realization of light matter interaction

$$H = \omega\hat{n} + \frac{\omega_0}{2}\sigma_z + g(a + a^\dagger)(\sigma_- + \sigma_+) = \begin{pmatrix} \omega\hat{n} + \frac{\omega_0}{2} & g(a+a^\dagger) \\ g(a+a^\dagger) & \omega\hat{n} - \frac{\omega_0}{2} \end{pmatrix}$$

Because the quantum Rabi model may be transformed to the ion-laser interaction, the optical realization may be extended to this case:

$$H = \nu\hat{n} + \frac{\omega_{21}}{2}\sigma_z + \lambda E_0\left[e^{i(kx-\omega t)}\sigma_+ + e^{-i(kx-\omega t)}\sigma_-\right]$$

By moving to a frame rotating at $\omega$ we get rid off the time dependence

$$H_{\text{ion}} = \nu\hat{n} + \frac{\delta}{2}\sigma_z + \Omega\left(\sigma_+\hat{D}(i\eta) + \sigma_-\hat{D}^\dagger(i\eta)\right)$$

$$D(i\eta) = e^{ikx} = e^{i\eta(a+a^\dagger)} = \sum_{m=0}^{\infty}\frac{(i\eta)^m}{m!}(a+a^\dagger)^m$$

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{D}^{\dagger}(\beta) & \hat{D}(\beta) \\ -\hat{D}^{\dagger}(\beta) & \hat{D}(\beta) \end{pmatrix}$$

$$\beta = i\,\eta/2$$

$$\mathcal{H}_{\text{ion}} = TH_{\text{ion}}T^{\dagger} = \begin{pmatrix} \nu\hat{n} + \Omega + \frac{\nu\eta^2}{4} & \frac{\iota\eta\nu}{2}\left(\hat{a} - \hat{a}^{\dagger}\right) + \frac{\delta}{2} \\ \frac{\iota\eta\nu}{2}\left(\hat{a} - \hat{a}^{\dagger}\right) + \frac{\delta}{2} & \nu\hat{n} - \Omega + \frac{\nu\eta^2}{4} \end{pmatrix}$$

# Eigenstates of the ion-laser interaction and/or the Rabi model can be obtained

$$\left|\psi_{\text{ion}}^{+}\right\rangle = |e\rangle \left( \frac{\Omega}{\nu} |0\rangle + \frac{i\eta\nu}{\Omega} |1\rangle \right) + |\mathrm{g}\rangle \, |-i\eta\rangle$$

$$|\psi\rangle = \frac{\Omega}{\nu} \sum_{n=0}^{m+1} c_n \, |n\rangle \, |e\rangle + \sum_{n=0}^{m} d_n \, |-i\eta, n\rangle \, |\mathrm{g}\rangle$$

H.M.-C., D. Jonathan and P.L. Knight,
Journal of Modern Optics 50, 265 (2003).
*A family of exact eigenstates for a single trapped ion interacting with a laser field.*

$$H = \omega\hat{n} + \frac{\omega_0}{2}\sigma_z + g(a + a^\dagger)(\sigma_- + \sigma_+) = \begin{pmatrix} \omega\hat{n} + \frac{\omega_0}{2} & g(a + a^\dagger) \\ g(a + a^\dagger) & \omega\hat{n} - \frac{\omega_0}{2} \end{pmatrix}$$

$$T = \frac{1}{2}\begin{pmatrix} (-1)^{\hat{n}} - 1 & (-1)^{\hat{n}} + 1 \\ -(-1)^{\hat{n}} - 1 & 1 - (-1)^{\hat{n}} \end{pmatrix}$$

$$H_T = THT^\dagger = \begin{pmatrix} \omega\hat{n} + \frac{\omega_0}{2}(-1)^{\hat{n}} - g(a + a^\dagger) & 0 \\ 0 & \omega\hat{n} - \frac{\omega_0}{2}(-1)^{\hat{n}} - g(a + a^\dagger) \end{pmatrix}$$

H. M.-C., A. Vidiella-Barranco, J. A. Roversi, S. M. Dutra,
Journal of Optics B 2, 21-23 (2000).
*Unitary transformation approach for the trapped ion dynamics.*

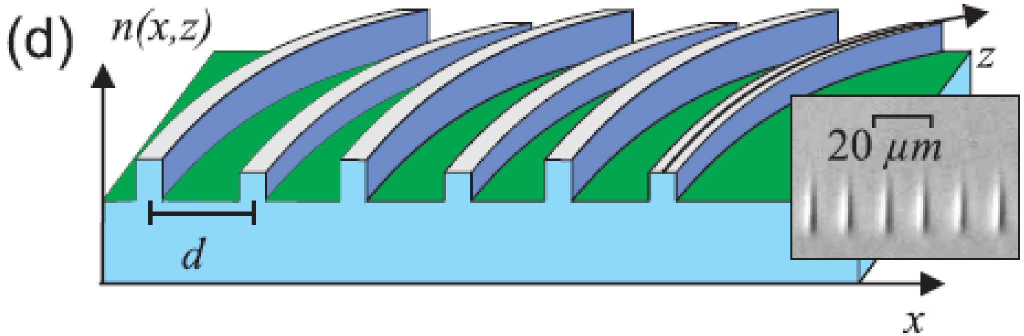
## Element "ee" of the diagonal matrix reads

$$H_e = \omega\hat{n} + \frac{\omega_0}{2}(-1)^{\hat{n}} - g(a + a^\dagger)$$

$$\left|\psi_e\right\rangle = \sum_{n=0}^{\infty} E_n(t)\left|n\right\rangle$$

$$i\frac{dE_0}{dZ} + \frac{\omega_0}{2}E_0 - gE_1 = 0,$$

$$i\frac{dE_n}{dZ} + \left(\omega n + \frac{\omega_0}{2}(-1)^n\right)E_n - g\sqrt{n+1}E_{n+1} - g\sqrt{n}E_{n-1} = 0, \quad n > 1$$



# Approximations done:

1. Optical rotating wave approximation
2. Removing time dependence
3. Vibrational wave approximation

$$H_{\text{ion}} = \nu\hat{n} + \frac{\delta}{2}\sigma_z + \Omega\left(\sigma_+\hat{D}(i\eta) + \sigma_-\hat{D}^\dagger(i\eta)\right)$$

$$D(i\eta) = e^{ikx} = e^{i\eta(a+a^\dagger)} = \sum_{m=0}^{\infty}\frac{(i\eta)^m}{m!}(a+a^\dagger)^m$$

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# Fast Quantum Rabi Model with Trapped Ions

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$$H = \begin{pmatrix} \nu\hat{n} & \Omega\hat{D}(i\eta) \\ \Omega\hat{D}^{\dagger}(i\eta) & \nu\hat{n} \end{pmatrix}. \tag{14}$$

We follow the unitary operator procedure introduced by Moya-Cessa *et al.*[23,24] and define the transformation matrix (whose elements are displacement operators)

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{D}^{\dagger}(i\eta/2) & \hat{D}(i\eta/2) \\ -\hat{D}^{\dagger}(i\eta/2) & \hat{D}(i\eta/2) \end{pmatrix}. \tag{15}$$

It is possible to check after some algebra that

$$\mathcal{H}_{\mathrm{QRM}} = THT^{\dagger} = \begin{pmatrix} \nu\hat{n} + \Omega + \frac{\nu\eta^2}{4} & \frac{\iota\eta\nu}{2}(\hat{a} - \hat{a}^{\dagger}) \\ \frac{\iota\eta\nu}{2}(\hat{a} - \hat{a}^{\dagger}) & \nu\hat{n} - \Omega + \frac{\nu\eta^2}{4} \end{pmatrix}, \tag{16}$$

that, after returning to the former (spin matrices) notation reads

$$\mathcal{H}_{\mathrm{QRM}} = \nu\hat{n} + \Omega\sigma_z + \frac{\iota\eta\nu}{2}(\sigma_+ + \sigma_-)(\hat{a} - \hat{a}^{\dagger}) + \frac{\nu\eta^2}{4}, \tag{17}$$

# Conclusions

It was shown that the Rabi model and the ion laser interaction Hamiltonians may be related via a simmilarity transformation.

This allows much faster regimes than the ones reached with the vibrational rotating wave approximation, as thta is dictated by the Rabi frequency (intensity of the laser).