Arithmetic Progressions and Symbolic Dynamical Systems

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van der Waerden’s Theorem

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Theorem: van der Waerden 1927

Suppose $\mathbb{N}$ is partitioned into two sets S_1 and S_2. Then either S_1 or S_2 has arbitrarily long arithmetic progressions — i.e. $\exists S_i$ such that for every $k \geq 2$, there are integers a and b such that have

$$a, \quad a + 2b, \quad \ldots, \quad a + (k-1)b \in S_i.$$

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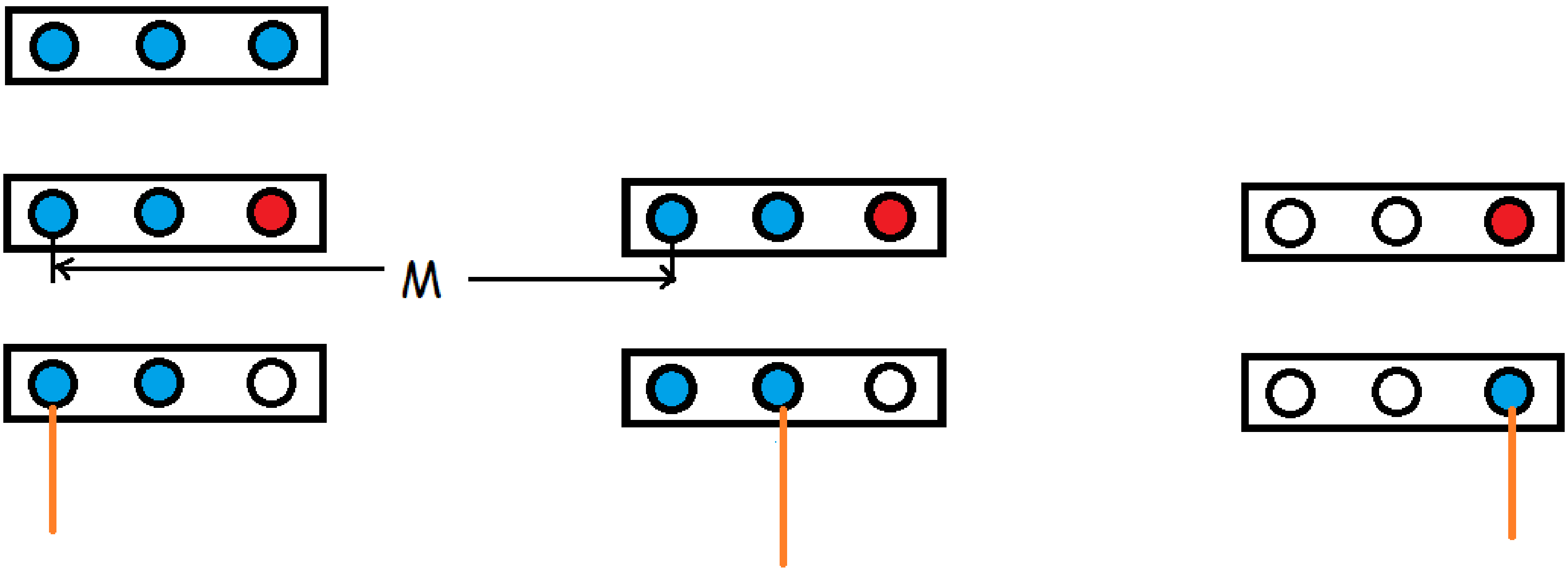
There are 32 possible block colorings. Pigeonhole $\implies$ 2 blocks in the first 33 are colored the same.

Proof of vdW



Erdős' Conjecture

Definition

A set $S \subseteq \mathbb{Z}$ has *positive upper Banach density* if

$$\limsup_{N\to\infty} \frac{|S \cap [-N, N]|}{2N+1} > 0.$$

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Theorem: (Szemerédi 1975)

Erdős conjecture holds.

Proof uses his "regularity lemma".

Some highlights

1. Roth 1956 Erdős Conjecture holds for length 3 A.P.
2. Szemerédi’s Theorem 1975
3. Furstenberg’s ergodic theory proof 1978
4. Gowers’ Fourier Analytic proof, 1996
5. Green-Tao A.P. in primes

Topological Dynamics

Connections between topological dynamics and integer sets.

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If X is a compact space and $T : X \to X$ is a continuous map, then (X, T) is said to be a *dynamical system*.

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Pigeonhole principle and Recurrence in Open Covers

Pigeonhole and Basic Recurrence

Theorem: The infinite pigeonhole principle

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Let (X, T) be a toplogical dynamical system, and $(U_\alpha)_{\alpha \in \Omega}$ be an open cover of X. Then there is a U_α in the cover for which for infinitely many n, $U_\alpha \cap T^n U_\alpha \neq \emptyset$.

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Pigeonhole Principle $\Rightarrow$ there is some U_i, $1 \leq i \leq n$ such that for infinitely many n, $T^n x \in U_i$.

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Consider $O = \{n \in \mathbb{Z} \mid T^n x \in U_i\}$. Pick some $n_0 \in O$.

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Consider $O = \{n \in \mathbb{Z} \mid T^n x \in U_i\}$. Pick some $n_0 \in O$.

$\forall n \in O$, $T^{n_0}x = T^{n_0-n}T^n x$. Hence $T^{n_0}x \in U_i \cap T^{n_0-n}U_i$.

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$\forall n \in O$, $\quad T^{n_0} x = T^{n_0 - n} T^n x$. Hence $T^{n_0} x \ \in \ U_i \cap T^{n_0 - n} U_i$.

Hence for infinitely many n, $U_i \cap T^{n_0 - n} U_i \neq \emptyset$.

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Consider the cover $(U_c)_{c \in \Omega}$ where U_c are the points in X_a with 0 colored c.

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By recurrence in open covers,

$$\exists c \in \Omega \;\; \exists^\infty n \;\; U_c \cap T^n U_c \neq \emptyset. \tag{1}$$

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$$\exists c \in \Omega \ \exists^{\infty} n \ \ U_c \cap T^n U_c \neq \emptyset. \qquad (1)$$

Since X_a is the orbit closure of a, there is a $k \in \mathbb{Z}$ such that $T^k a \in U_c \cap T^n U_c$. That is, $a_{-k} = c$ and $a_{-k+n} = c$. This is true for all n in (1).

van der Waerden's Theorem and Multiple Recurrence in Open Covers

The version of recurrence in tds which is equivalent to Van der Waerden's theorem is the following.

Theorem: Multiple Recurrence in Open Covers

Let (X, T) be a topological dynamical system and $(U_\alpha)_{\alpha\in\Omega}$ be an open cover of X. Then there is a U_α in the cover such that

$$\forall k \geq 2 \ \exists n > 0 \qquad U_\alpha \cap T^n U_\alpha \cap \cdots \cap T^{(k-1)n} U_\alpha \neq \emptyset.$$

Szemerédi’s Theorem and Furstenberg Multiple Recurrence Theorem

Dynamical Systems view of Szemerédi's Theorem

For Szemerédi's theorem, we now have to consider *measure* as well.

Definition

A *measure-preserving topological dynamical system* is a quadruple $(X, \mathcal{X}, \mu, T)$ is a space where

- X is a compact topological space,
- $\mathcal{X}$ is a σ-algebra on X,
- μ a probability measure on $\mathcal{X}$ and
- $T : X \to X$ is a measure-preserving homeomorphism.

Multiple Recurrence

Theorem: Multiple Recurrence Theorem

Let $(X, \mathcal{X}, \mu, T)$ be a mtds. Then for any $E \in \mathcal{X}$ with $\mu(E) > 0$, we have

$$\mu(E \cap T^n E \cap \cdots \cap T^{(k-1)n} E) > 0.$$

Furstenberg Correspondence Principle

Lemma

Let $(X, \mathcal{X}, \mu, T)$ be as in the FMRT, and E have positive measure. Then there is an F, $\mu(F) > 0$ such that for every x in F,

$$\{n \in \mathbb{Z} \mid T^n x \in E\}$$

has positive upper density.

Proof of Lemma

Proof.

- Define $\delta_N(x)$ to be the frequency with which $T^{-N}x$, $\ldots$, $T^N x$ visits E. Then the expected value of δ_N is $\mu(E)$.
- The probability of

$$A_N = \left\{ x \in X \mid \delta_N(x) \geq \frac{1}{2}\mu(E) \right\}$$

  is at least $1/2\mu(E)$.
- Then F is the set $\bigcap_N \bigcup_{m>N} A_m$.

$\square$

Effective Versions

Furstenberg Multiple Recurrence Theorem - Pointwise

Theorem: (Pointwise)

Let $(X, \mathcal{B}, \mu)$ be a probability space. Let T be a measure-preserving operator. Let $A \in \mathcal{B}$ with $\mu(A) > 0$.
Then $\forall k$, for μ-a.e. $x \in A$,

$$\exists n \;\; x \in T^{-n}(A) \wedge x \in T^{-2n}A \wedge \cdots \wedge x \in T^{-kn}A.$$

Effective Versions - Cantor Space, Left Shift

Definition

We say that a point X in Cantor Space is *k-recurrent* in $P \in \mathcal{B}$ if $\exists n \geq 1$ such that $X \in \cap_{i=1}^{k} X \in T^{-in}(P)$.

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Location (P)	Randomness Notion (X)
clopen	Kurtz Randomness
Π_1^0 with eff. positive measure	Schnorr Randomness
non-null Π_1^0	Martin-Löf randomness

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Theorem

If P is a non-empty clopen set, then every Kurtz random X is multiply recurrent in P.

Proof. (Sketch) Suppose every string in P is shorter than N bits.
Let $n_0 = N$, $n_1 = (k+1)n_0$, $n_2 = (k+1)n_1$, $\ldots$.
Test:

$$Q = \bigcap_{t \in \mathbb{N}} \{Y \mid \exists i \in [1,k] \;\; Y_{in_t} \notin P\}.$$

□

Effectively Positive Π^0_1 sets and Schnorr randoms

Theorem

Let $P \in 2^{\mathbb{N}}$ be a Π^0_1 class with a computable positive measure $\lambda(P)$. Then each Schnorr random is multiply recurrent in P.

(Proof)

- Let $B = 2^N - P = \cup_s B_s$, an effectively open set.
- At any finite stage s, we check X is multiply recurrent in $2^{\mathbb{N}} - B_s$.
  - Let $n_t \geq n_{t-1}(k+1)$ be so large that
  $$\lambda(B - B_{n_t}) \leq 2^{-(t+v+k)}.$$
  - Q_v is the set of all sequences Z with at least one of $2^{n_t}Z$, $2^{2n_t}Z$, $\ldots$, $2^{kn_t}Z$ in B_{n_t}. (hence non-k-recurrent in P).
  - This set Q_v is Π^0_1 and null. Hence if $Z \in Q_v$, then Z is Kurtz-non-random, hence Schnorr-non-random.
- $\ldots$

Proof (continued)

- ...
  - The error class for v at stage t is

$$G_v^t = \{Y \mid \exists i \in [1,k] \quad Y_{in_t} \in (B - B_{n_t})\}$$

  - Then $\lambda(G_v^t) \leq k2^{-(t+v+k)}$, by the union bound and computable.
  - Then $G_v = \cup_t G_v^t$ has probability less than 2^{-v}.
  - Hence if Z is Schnorr-random, then there is a G_v excluding Z. Hence Z is multiply recurrent in P.

Positive Π^0_1 sets and Martin-Löf randoms

Theorem

Let $P \in 2^{\mathbb{N}}$ be a Π^0_1 class with measure $\lambda(P) > 0$. Then each Martin-Löf random is multiply recurrent in P.

Effective Versions - Kronecker Systems

Definition

Let G be a compact group, and for some $a \in G$, define $T_a : G \to G$ by $T_a(x) = a \cdot x$. Then (G, T_a) is called a *Kronecker System*.

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Since G is a group, if there is any recurrent point in it, then every point in it must be recurrent.

Theorem

Every point in a Kronecker System is multiply recurrent.

Thank You!