Birkhoff's ergodic theorem for measure-preserving transformations: the harder part

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Definition

$(\Omega, \mathcal{B}, \mu, T)$ is a *dynamical system* if

- $(\Omega, \mathcal{B}, \mu)$ is a probability measure space, and
- $T : \Omega \to \Omega$ is a function so that for every $B \in \mathcal{B}$, $T^{-1}(B)$ is measurable and $\mu(T^{-1}(B)) = \mu(B)$.

When $f : \Omega \to \mathbb{R}$, the *ergodic averages* are the functions

$$(A_N f)(x) = \frac{1}{N} \sum_{i=0}^{N-1} f(T^i x).$$

Theorem (Birkhoff's Pointwise Ergodic Theorem)

Suppose $(\Omega, \mathcal{B}, \mu, T)$ *is a dynamical system and* $f : \Omega \to \mathbb{R}$ *is an* L_1 *function. Then for almost every* $x \in \Omega$*, the limit*

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Definition

We say x is *weak Birkhoff* for $(\Omega, \mathcal{B}, \mu, T)$ and f if $\lim_{N\to\infty}(A_N f)(x)$ exists.

Question

Under computability assumptions on $(\Omega, \mathcal{B}, \mu, T)$ and f, which points are weak Birkhoff?

In this talk:

- $\Omega = 2^{\mathbb{N}}$ is always Cantor space.
- $\mathcal{B}$ is the σ-algebra generated by the basic clopen sets: when $\sigma : [0, n-1] \to \{0, 1\}$ is a finite sequence,

$$[\sigma] = \{\omega \in \Omega \mid \omega \upharpoonright n = \sigma\}$$

  is a *basic clopen set*.
- the measure is always given by $\mu([\sigma]) = 2^{-|\sigma|}$,
- T is always computable.

Theorem (V'yugin)

If f is computable and x is Martin-Löf random then x is weak Birkhoff.

Theorem (F.-T.)

If x is not Martin-Löf random then there is a T and a computable f so that x is not weak Birkhoff.

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Theorem (Miyabe-Nies-Zhang)

If f is lower semi-computable and x is OW-random then x is weak Birkhoff.

Definition

When $\alpha < \beta$, we say that x has k $\alpha - \beta$-*upcrossings* if there is a sequence

$$i_0 < j_0 < i_1 < j_1 < \cdots < i_{k-1} < j_{k-1}$$

such that for each $n \leq k$,

- $(A_{i_n} f)(x) > \beta$, and
- $(A_{j_n} f)(x) < \alpha$.

We are given a Martin-Löf test $\{U_i\}$. We partition $\Omega = A \cup B \cup C$.

- We only try to create upcrossings for points $x \in A \cap \bigcap_i [U_i]$.
- $f = \chi_B$: we create upcrossings by ensuring that $\{T^i x, T^{i+1} x, \ldots, T^j x\} \subseteq B$ for long intervals $[i, j]$.
- C is the "cool-down" space: we complete an upcrossing by ensuring that $\{T^{j+1} x, \ldots, T^k x\} \subseteq C$ for long intervals $[j+1, k]$.

We construct the transformation T stage by stage as we enumerate the sets U_i.

We need a way to give partial information about the transformation T. A partial transformation T_n should specify, for each $\sigma \in 2^{<\omega}$, a $T_n(\sigma) \in 2^{<\omega}$, meaning that $T([\sigma]) \subseteq [T_n(\sigma)]$.

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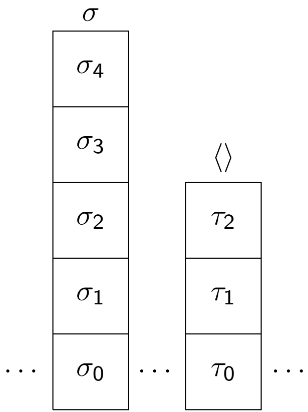
We make the following assumption: at stage n, there is a value k so that, for each $\sigma \in 2^k$, either:

- $|T_n(\sigma)| < |\sigma|$ and, for all $\tau \sqsupseteq \sigma$, $T_n(\tau) = T_n(\sigma)$ (i.e. we have not fully specified what the behavior of T will be on $[\sigma]$, and have made no decisions on longer sequences), or
- $|T_n(\sigma)| = |\sigma|$, and for all ρ, $T_n(\sigma^\frown\rho) = T_n(\sigma)^\frown\rho$ (i.e. we have specified exactly where $[\sigma]$ goes, have also determined the behavior on extensions of σ, and the behavior on extensions is very simple).

To describe these transformations, we worked with "tower diagrams":



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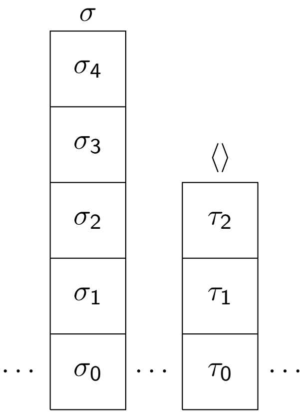

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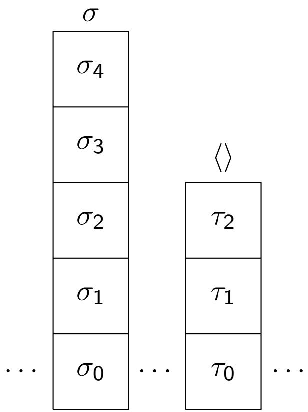

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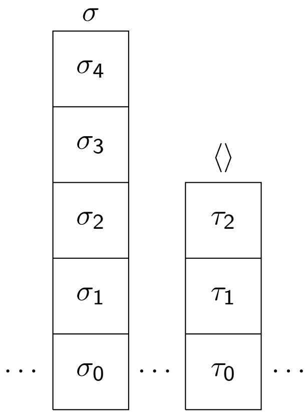
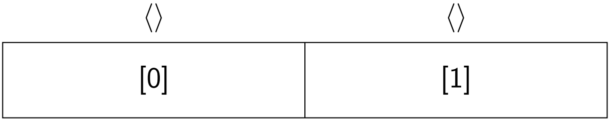


- The diagram is read bottom up: $T_n(\sigma_0) = \sigma_1$, etc.
- Stacked boxes indicate "perfect fits": $|\sigma_0| = |\sigma_1|$.
- The notation above a top box indicates partial information: $|\sigma| < |\sigma_4|$,
- The notation on top always has to allow the possibility of a loop: $\sigma \sqsubset \sigma_0$.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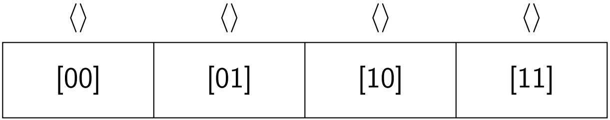
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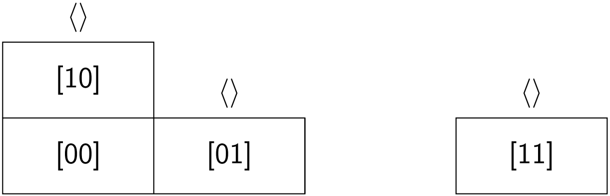
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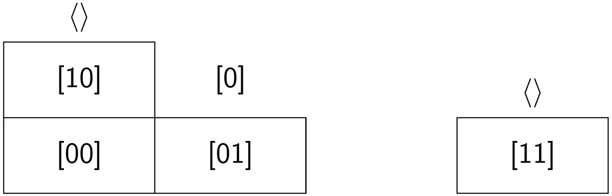
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We build a sequence of partial transformations T_n that extend each other with limit T.

At alternate stages, we'll act to ensure that

- the limit T of this sequence is a transformation except on an effective F_σ set with measure 0 and
- $\lim_{N\to\infty}(A_N f)(x)$ does not exist for any x in (a particular subset of) the intersection of a universal Martin-Löf test.

Outrun

Thinning Loops Lemma

We can extend a partial transformation with an open loop to one in which this open loop is replaced by one of the same measure but arbitrarily small width. In fact, we can do so in a way that guarantees that some fraction of the open loop remains untouched.

Outlast

Escape Lemma

If σ_0 appears in an open loop in T, we can extend T to a new partial transformation where $[\sigma_0]$ is contained in an open loop with $\langle\rangle$ on top.

Cutting and stacking

Build T to be the limit of finite partial transformations T_n.

- Put aside portions of B to guarantee that $(A_{n_1}\chi_B)(x) \geq \frac{1}{2}$ for some n_1 and portions of C to guarantee that $(A_{n_2}\chi_B)(x) \leq \frac{1}{3}$ for a larger n_2.
- If we see some σ enter V_i, we map σ through a portion of B repeatedly, then through a portion of C.

This will ensure that there are infinitely many upcrossings for any $x \in \cap_i [V_i]$.

The 26 inductive conditions

1–12 Structure of components. Ensure that each partial transformation is extendible to another of the right kind.

13–22 Upcrossing management.

23–26 Totality management.

Lower semicomputable f

Theorem (F.-T.)

If f is lower semi-computable and x is weakly 2-random then x is weak Birkhoff.

Later, this was improved to:

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$$\mathrm{W2R} \subset \mathrm{OW} \subseteq \mathrm{DensityR} \subset \mathrm{DiffR} \subset \mathrm{ML}$$

- Difference randomness: A real x is difference random if it is in the intersection of every difference test: a sequence $\{U_i, V_i\}$ such that $\mu([U_i] - [V_i]) \leq 2^{-i}$

Because we are dealing with a computably enumerable function, creating the high points of upcrossings is (fairly) easy: we can simply increase the value of the function on a certain interval to make the ergodic average go up.

But to bring the ergodic average back down, we need many points where the value of the function is small. This is a limited resource we have to manage carefully.

The following situation seems inevitable:

- A segment $[\sigma]$ gets enumerated into some U_i.
- We arrange for every point in $[\sigma]$ to get an upcrossing.
- Many stages pass. Some points in $[\sigma]$ are allocated additional upcrossings.
- Some $[\tau] \subseteq [\sigma]$ is added to V_i and some new $[\tau']$ is enumerated into U_i. We need to reclaim the measure used to give $[\tau]$ an upcrossing to make a new upcrossing somewhere else.
- But $[\tau]$ is in the middle of a stack of upcrossings which remain valid.
- Because of the recursion theorem, this situation seems unavoidable: under even weak uniformity assumptions, an opponent can wait until this situation happens and then enumerate a τ which it knows to cause this problem.