A non-commutative Mrówka's Ψ-space

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Definition

A sequence $(a_{\beta,\alpha} : \alpha, \beta \in \kappa)$ of noncompact elements of $\mathcal{B}(\ell_2)$ is called a "system of almost matrix units" if it satisfies for each $\alpha, \beta, \xi, \eta < \kappa$:

- $(a_{\beta,\alpha})^* =^* a_{\alpha,\beta}$,
- $a_{\eta,\xi} a_{\beta,\alpha} =^* \delta_{\xi,\beta} a_{\eta,\alpha}$,

where $a =^* b$ means $a - b \in \mathcal{K}(\ell_2)$.

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For each pair $(\xi, \eta) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ we associate an operator on $\ell_2(2^{<\mathbb{N}})$

$$T_{\eta,\xi}(s) = \begin{cases} e_{\eta|k} & \text{if } s = e_{\xi|k} \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

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Lemma

Let $R \in \mathcal{B}(\mathcal{H})$ and U be a Borel subset of $\mathbb{C}$, then the set

$$B_U^R = \{(\eta, \xi) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : T_{\eta,\eta} R T_{\xi,\xi} =^* \lambda T_{\eta,\xi},\ \lambda \in U\}$$

is Borel in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. In particular, if B_U^R is either countable or of size of the continuum.

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If P, Q are projections, then PQ is not a projection unless P and Q commute.