basis problem and related properties

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Basic Questions

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What properties are closely related to the basis problem?

The real line and the Sorgenfrey line

Theorem (Baumgartner 1973)

PFA implies that every set of reals of cardinality $\aleph_1$ *embeds homomorphically into any uncountable regular space of countable network and that*

every subset of the Sorgenfrey line $(\mathbb{R}, \rightarrow)$ *of cardinality* $\aleph_1$ *embeds homomorphically into any uncountable subspace of* $(\mathbb{R}, \rightarrow)$*.*

S and L spaces

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An L space is a regular hereditarily Lindelöf (HL) space which is not separable.

S space and HL

Theorem (M.E. Rudin, 1972)

It is consistent to have an S space.

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So under PFA, an uncountable regular space either contains an uncountable discrete space or is HL.

L space

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It turns out that the class of L spaces does not have a reasonably small basis.

Algebraic operation

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Also, the class of L groups does not have a reasonably small basis.

first countable

Theorem (Szentmiklossy, 1980)

MA_{ω_1} implies that there are no first countable L spaces.

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Question

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Question

Is it consistent that every perfectly normal locally connected compact space is metrizable?

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If X and Y are compact and $X \times Y$ is perfectly normal, must one of X and Y be metrizable?

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If there is a counterexample to the basis problem (for any class closed by adding countably many open sets), then there is a submetrizable one.

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Question (PFA)

Is there a property that contains 2 HL elements and is preserved under continuous image?

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(Gruenhage) What about first countable + HL?

(Kunen) What about spaces with HL compactification?

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Proposition (PFA)

If a first countable HL space X is not submetrizable, then there are $\{(x_\alpha, y_\alpha) \in X^2 \setminus \Delta : \alpha < \omega_1\}$ and local bases $\{u_{x_\alpha,n}, u_{y_\alpha,n} : \alpha < \omega_1, n < \omega\}$ such that $u_{x_\alpha,n}$ ($u_{y_\alpha,n}$) splits only one pair.

Cometrizable

A topological space X is cometrizable if it has a weaker metrizable topology and a neighbourhood assignment consisting of closed sets in this weaker topology.

Theorem (Gruenhage 1987)

Assume PFA. A cometrizable space has a countable network if it contains no uncountable discrete subspace nor an uncountable subspace of the Sorgenfrey line.

Inner topology

Definition

For a topological space (X, τ) *and a collection* $\mathcal{C} \subset P(X)$, *the inner topology* $(X, \tau^{I,\mathcal{C}})$ *induced by* $\mathcal{C}$ *is the topology with base* $\{\{x\} \cup O^{I,\mathcal{C}} : x \in O, O \text{ is open}\}$ *where* $O^{I,\mathcal{C}} = \cup\{C \in \mathcal{C} : C \subset O\}$.

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Theorem (PFA)

If (X, τ) is regular and $(X, \tau^{I,\mathcal{C}})$ is HL for some countable $\mathcal{C}$, then (X, τ) either has a countable network or contains an uncountable Sorgenfrey subset.

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Proposition (PFA)

If X is first countable, regular and contains no uncountable separable metrizable or Sorgenfrey subset, then for any countable collection $\mathcal{C}$, $(X, \tau^{I,\mathcal{C}})$ is a countable union of discrete subsets.

Outer "topology"

Definition

For a topological space (X, τ) *and a collection* $\mathcal{C} \subset P(X)$, *the outer "topology"* $(X, \tau^{O,\mathcal{C}})$ *induced by* $\mathcal{C}$ *is the collection* $\{O^{O,\mathcal{C}} : O$ *is open*$\}$ *where* $O^{O,\mathcal{C}} = \cap\{C \in \mathcal{C} : O \subset C\}$.

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Suppose X is a regular, HL space. Any outer topology induced by a countable collection either has a countable network or contains an uncountable Sorgenfrey subset.

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Example. Cometrizable spaces.

Outer "topology" to covering property

Proposition (PFA)

Suppose X is a first countable regular, HL space, $\mathcal{C}$ is countable such that $(X, \tau^{O,\mathcal{C}})$ is metrizable and $(X, \langle \{x\} \cup (u^{O,\mathcal{C}}_{x,n} \setminus u_{x,n}) : x \in X \rangle)$ contains no uncountable HL subset for all n. Then X contains an uncountable metrizable subset.

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Proposition (PFA)

Suppose X is a first countable regular, HL space, $\mathcal{C}$ is countable such that $(X, \tau^{O,\mathcal{C}})$ is Sorgenfrey and $(X, \langle \{x\} \cup (u^{O,\mathcal{C}}_{x,n} \setminus u_{x,n}) : x \in X \rangle)$ contains no uncountable HL subset for all n. Then for any $Y \in [X]^{\omega_1}$ and $n < \omega$, there is $Y' \in [Y]^{\omega_1}$ such that

$[x, \infty) \cap Y' \subset u_{x,n}$ for all $x \in Y'$.

Covering property

People have considered to force properties of X from covering properties of its finite powers.

Fact (MA_{ω_1})

Suppose that X is a first countable space with covering property (**): for any $m, n < \omega$, for any $\{a_\alpha \in X^n : \alpha < \omega_1\}$, there are $\alpha \neq \beta$ such that for any $i < n$, $a_\alpha(i) \in u_{a_\beta(i),m}$ and $a_\beta(i) \in u_{a_\alpha(i),m}$. Then X contains a metrizable subspace.

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Question

Is it consistent that X has an uncountable metrizable subspace if X^ω is HL?

A weaker covering property

Definition

A first countable space X with a real ordering $<$ has property $(*)$ if

for any $n < \omega$, for any $(m_0, ..., m_{n-1}) \in \omega^n$, for any $\{a_\alpha, b_\alpha \in X^n : \alpha < \omega_1\}$ such that $b_\alpha(i) \in u_{a_\alpha(i),m_i} \cap (a_\alpha(i), \infty)$ whenever $\alpha < \omega_1, i < n$, there are $\alpha \neq \beta$ such that for any $i < n$, $b_\alpha(i) \in u_{a_\beta(i),m_i}$ and $b_\beta(i) \in u_{a_\alpha(i),m_i}$.

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Theorem (PFA)

Assume that X is a first countable regular space with property $(*)$ and X has no uncountable left sub-Sorgenfrey subspace. Then X contains an uncountable metrizable or Sorgenfrey subspace.

Thank you!