Black Holes in $f(R)$ gravity: scalar hair or absence thereof ?

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in collaboration with

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Outline

In this talk I will show first how solutions with *trivial* Ricci scalar, i.e., $R = const$ emerge in $f(R)$ gravity. These solutions include the best known BH's solutions found in GR, like the *asymptotically flat* (AF) (vacuum) BH's solutions (i.e. $R = 0$) and those endowed with an effective cosmological constant *asymptotically (anti) De Sitter* ones (ADS or AADS).

Then (and this is the main goal of this talk) I will show both *analytical* (i.e. when No-Hair Theorems do apply) and *numerical* (i.e. when NHT's do not apply) evidence about the absence of *geometric* (scalar) hair in *static and spherically symmetric and AF BH* (SSSAFBH) solutions for several $f(R)$ models.

By *geometric* hair I mean SSSAFBH *regular* solutions where $R(r)$ interpolates non-trivially from the event horizon r_h to spatial infinity $r = \infty$.

The conclusion is that within the framework of some classes of $f(R)$ models proposed for several physically-motivated purposes (e.g. late accelerated expansion, inflation, etc), the only SSSAFBH regular solution seems to be Schwarzschild's where $R(r) = 0$.

Remark: here by *regular* is meant a BH with a regular horizon and with no singularities in the domain $r \geq r_h$ (i.e. no naked singularities).

Some of our papers on $f(R)$:

On cosmology:

– L. Jaime, L. Patiño, M. Salgado, PRD, **87**, 024029, (2013)
– *idem*, PRD, **89**, 084010, (2014)
– *idem*, arXiv:1206.1642 (a thorough review)
– *idem*, in Relativity and Gravitation, 100 Years After Einstein in Prague, Springer, Heidelberg, 2014; arXiv:1211.0015

On star-like objects:

– *idem*, PRD, **83**, 024039, (2011)

On black-holes (current presentation):

L. Jaime, P. Cañate, M. Salgado: arXiv: 1509.01664. CQG (in press)

$f(R)$ GRAVITY (BRIEF HISTORICAL REMARKS)

– Non-linear Lagrangians in R, R_{ab}, and R_{abcd} date back since the years that followed GR (H. Weyl, 1921; K. Lanczos, 1938; Buchdahl 1970). They were analyzed much later in different contexts. For instance, in cosmology ...

– 1979 (A. Starobinsky), as models for *inflation.*: $f(R) = R - aR^2$.

– 1982 (R. Kerner) as a "cosmological model without singularity". Remark: Several $f(R)$ models considered today are very similar to those considered in that paper.

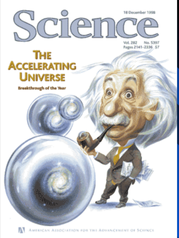
– 1986 (J.P. Durisseau & R. Kerner) as a "reconstruction of inflationary model".

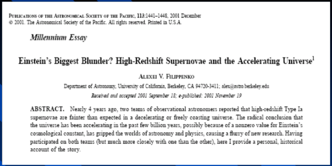
– As mentioned before, the discovery of the accelerated expansion of the Universe renewed the interest in this kind of models. The first ones proposed within the specific goal of producing an accelerated expansion were: Cappozziello (2002), Cappozziello et al. (2003), Carroll et al. (2004,2005).

– Since 2003 a boom of papers analyzing $f(R)$ gravity in all possible scenarios have appeared in the literature: perhaps more than 1000 papers ! ($\approx 2/week$).

SNIa data within the ΛCDM paradigm

$$D_L(z) = cH_0^{-1}(z+1)\int_0^z \frac{dz'}{H(z')} \text{ (for } k=0), \quad \mu = 5\log(D_L/Mpc) + 25 \ .$$





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Millennium Essay

Einstein's Biggest Blunder? High-Redshift Supernovae and the Accelerating Universe[1]

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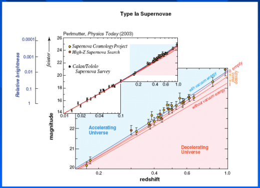
Received and accepted 2001 September 18; e-published: 2001 November 19

ABSTRACT. Nearly 4 years ago, two teams of observational astronomers reported that high-redshift Type Ia supernovae are fainter than expected in a decelerating or freely coasting universe. The radical conclusion that the universe has been accelerating in the past few billion years, possibly because of a nonzero value for Einstein's cosmological constant, has gripped the worlds of astronomy and physics, causing a flurry of new research. Having participated on both teams (but much more closely with one than the other), here I provide a personal, historical account of the story.



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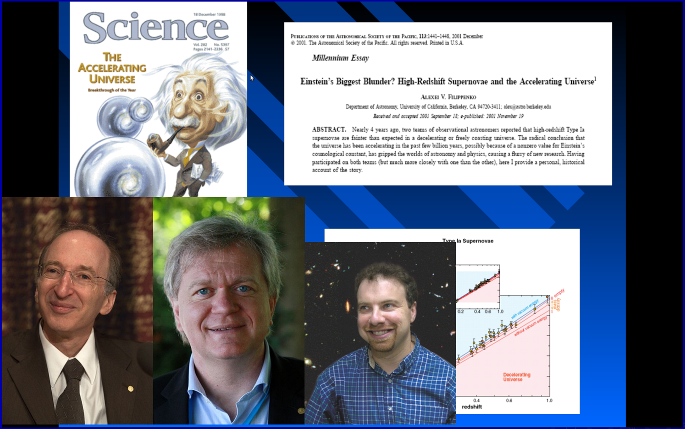

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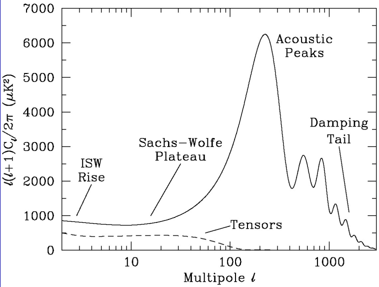

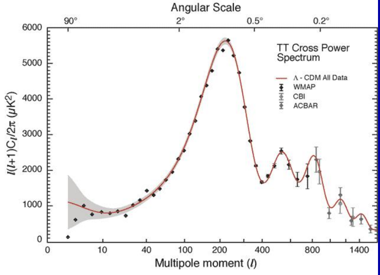

(from D. Scott and G. Smoot, arXiv/astro-ph/0406567 and WMAP team, respectively)

Motivation

– ΛDM paradigm within GR: the simplest and perhaps most successful cosmological model: reproduce successfully the SNIa and CMB observations...But still we don't know what DM is.

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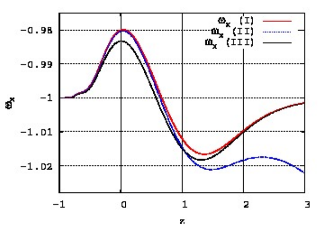
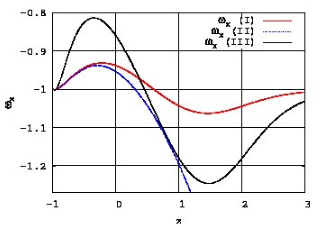
– $f(R)$ metric theories of gravity: a possible explanation for the accelerated expansion of the Universe without introducing new fields (As far as we know, DM **must** be considered, otherwise it seems impossible to recover the rest of cosmological observations.). These alternative theories of gravity (like others) allows for an "EOS of (geometric) dark energy" that varies in cosmic time, unlike Λ.

Variable EOS within $f(R)$ FRW cosmology (see L.Jaime, L. Patiño, MS, PRD **89**,084010,2014; PRD **87**,024029,2013;arXiv/1211.0015)

$$f(R)_{HS} = R - m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1} \qquad f(R)_{\rm St} = R + \lambda R_S \left[\left(1 + \frac{R^2}{R_S^2}\right)^{-q} - 1 \right]$$



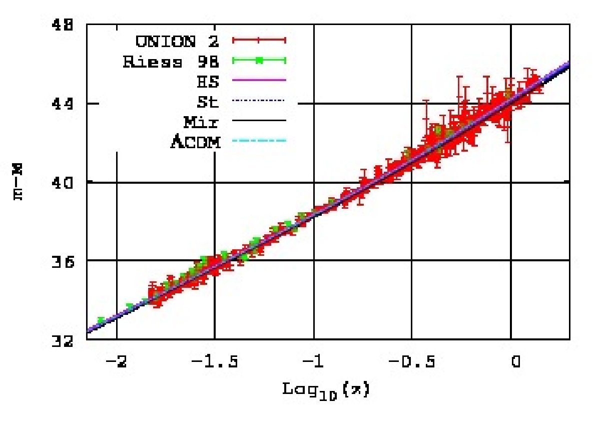


$$\omega_X = \frac{p_X}{\rho_X} \ , \ \rho_X = \frac{1}{\kappa f_R} \left\{ \frac{1}{2}(f_R R - f) - 3 f_{RR} H \dot{R} + \kappa \rho (1 - f_R) \right\} \ , \quad \dot{R} = \frac{dR}{dt} \ ,$$

$$p_X = -\frac{1}{3\kappa f_R} \left[\frac{1}{2}(f_R R + f) + 3 f_{RR} H \dot{R} - \kappa(\rho - 3 p_{\rm rad} f_R) \right] \ , \ f_R = \frac{df}{dR} \ , \ \kappa = 8\pi G_0$$

SNIa data within $f(R)CDM$ gravity (see L.Jaime, L. Patiño, MS, arXiv/1211.0015)

$D_L(z) = cH_0^{-1}(z+1)\int_0^z \frac{dz'}{H(z')}$ (for $k=0$), $\mu = 5\log(D_L/Mpc) + 25$.



CMB Power spectrum witin the $f(R)CDM$ gravity

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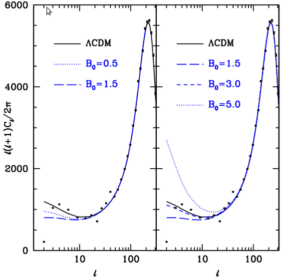

FIG. 2 (color online). CMB angular power spectrum C_ℓ for $f(R)$ models with the Compton wavelength parameter $B_0 = 0$ (ΛCDM), 0.5, 1.5, 3.0, 5.0. As B_0 increases, the ISW contributions to the low multipoles decrease, change sign, and then increase. WMAP3 data with noise error bars are overplotted and rule out $B_0 \geq 4.3$ (95% CL).

then become colder along directions associated with over-dense regions.

Figure 2 illustrates the impact of this effect on the CMB power spectrum. As the Compton wavelength approaches the Gpc ($k \sim 10^{-3}$ Mpc^{-1}) scales associated with the low multipoles of the ISW effect, the reduction in the decay of the potential also suppresses the ISW effect. Near $B_0 \approx 1.5$ the ISW effect is almost entirely absent at the quadrupole. Reduction in the amplitude of the quadrupole is, in fact, weakly favored by the data but the large cosmic variance of the low multipoles prevents this from being a significant improvement. Moreover, the elimination of the ISW effect at higher multipoles where the observed power is higher counteracts this improvement. For $B_0 \gtrsim 1.5$ the Compton wavelength exceeds the scales of interest and potential decay turns into potential growth. By $B_0 \sim 3$ the ISW effect has an equal amplitude to that of ΛCDM. For $B_0 \gtrsim 5$, it is too large to accommodate the WMAP data. These features drive the overall joint constraint on B_0 shown in Fig. 3. The WMAP data also serve to fix the parameters that control the high redshift universe. In particular, they constrain the initial amplitude of power on scales that are observed in galaxy surveys such as the SDSS LRG survey.

B. Galaxy power spectrum

The time-dependent Compton wavelength of $f(R)$ models induces a more dramatic effect in the matter power

$f(R)$ METRIC GRAVITY

$$I_{\rm JF}[g_{ab},\psi] = \int \frac{f(R)}{2\kappa}\sqrt{-g}\; d^4x + I_{\rm JF\,,\,matt}[g_{ab},\psi] \;, \tag{1}$$

where R =Ricci scalar, $f(R)$ is a C^3 but otherwise *an apriori* arbitrary function of R, $\kappa := 8\pi G_0$, and ψ represents collectively the matter fields – ordinary and DM – (here $c = 1$). Necessary conditions for an $f(R)$ model gravity to be free of pathologies are: $f_{RR} > 0$ and $f_R > 0$ (cf. equations below)
Varying the action Eq. (1) with respect to the metric yields

$$f_R R_{ab} - \frac{1}{2} f g_{ab} - (\nabla_a \nabla_b - g_{ab}\Box)\, f_R = \kappa T_{ab} \;, \tag{2}$$

where $f_R := \partial_R f$, $\Box = g^{ab}\nabla_a\nabla_b$, and T_{ab} is the EMT of matter.

THEOREM (EXERCISE)

Take ∇^a on both sides of Eq. (2) and prove that the EMT of matter is conserved:

$$\nabla^a T_{ab} = 0 \;\; . \tag{3}$$

back

When taking the trace of Eq. (2) we obtain the following EOM for R:

$$\Box R = \frac{1}{3f_{RR}}\left[\kappa T - 3f_{RRR}(\nabla R)^2 + 2f - Rf_R\right] , \tag{4}$$

where $(\nabla R)^2 := g^{ab}(\nabla_a R)(\nabla_b R)$, $T := T^a{}_a$. When expanding the derivative ∇ acting on f_R in (2), following the use of (4) we obtain

$$G_{ab} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\Big(Rf_R + f + 2\kappa T\Big) + \kappa T_{ab}\right] . \tag{5}$$

We shall be dealing with equations (4) and (5), and treat the theory as a system of second order coupled PDE for the Ricci scalar R and the metric g_{ab}, respectively.

Exercise: take $f(R) = R - 2\Lambda$ and show that Eqs. (5) and (4) reduce respectively to GR+Λ :

$$G_{ab} + g_{ab}\Lambda = \kappa T_{ab} \ , \tag{6}$$

$$R = 4\Lambda - \kappa T \ . \tag{7}$$

GR vs. $f(R)$ gravity

Not surprisingly the basic Axioms 1–6 of GR (associated with the metric and the manifold) are kept in $f(R)$ gravity (after all GR is a particular case of $f(R)$ gravity). However, the Axiom 7 below (associated with EOM) is not

1. The spacetime is a 4-dimensional differential manifold endowed with a Lorentzian metric (M, g_{ab}).
2. Gravitation is described geometrically in terms of spacetime curvature (R_{abcd}) ($R_{abcd} = 0$ only when the spacetieme is globally flat).
3. The theory should be covariant (diffeomorphism invariant).
4. The equivalence principle holds: test (point) particles move on geodesics of the metric g_{ab}. The laws of physics (those compatible with special relativity) are still valid locally.
5. The only quantity pertaining to spacetime that should appear in the laws of physics is the metric (*minimal coupling*).
6. Assume the usual Levi-Civita connection (no torsion and the theory is metric compatible $\nabla_a g_{ab} = 0$).
7. The field equations should be linear in the second derivatives (quasilinear PDE).
$f(R)$ theories keep all this axioms except "7": only fulfilled when $f(R) = R - 2\Lambda$.

Gravitational tests

Important tests for any gravitational theory:

- **Cosmology:** SNIa data, age of the Universe, nucleosynthesis, perturbed FRW (CMB), etc.
- **Solar system:** classical tests: Does $f(R)$ gravity really pass those tests or not ?
- **Strong gravity:** binary pulsar, neutron stars (mass vs. radius, EOS), BH's etc.
- **Formal issues:** Cauchy problem, singularity theorems, BH's: existence and uniqueness, etc.

As mentioned in the previous slide, hundreds of papers have been devoted to analyze $f(R)$ gravity in the the above and other scenarios. In this talk I will focus only on BH's.

$f(R)$ gravity can reduce to GRΛ $\to$ BH's with "trivial" R

Even if $f(R) \neq R - 2\Lambda$ the whole theory can reduce to GRΛ:
Suppose the EMT is traceless $T = 0$. Then $R = R_1 = const$ is a solution of

$$\Box R = \frac{1}{3f_{RR}}\left[\kappa T^{=0} - 3f_{RRR}(\nabla R)^2 + 2f - Rf_R\right] , \quad (8)$$

provided R is a root of $V'(R) = 2f - Rf_R$ (i.e. $2f(R_1) = R_1 f_R(R_1)$), assuming for instance $0 < f_{RR}(R_1) < \infty$.
That is, R_1 is an extremum (e.g. maximum or minimum) of the "potential" $V(R)$. In such an instance, and if $0 < f_R(R_1) < \infty$, the field equation

$$G_{ab} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\left(Rf_R + f + 2\kappa T^{=0}\right) + \kappa T_{ab}\right] , \quad (9)$$

reduces to

$$G_{ab} = -g_{ab}\Lambda_{\rm eff} + 8\pi G_{\rm eff} T_{ab} , \quad (10)$$

$$\Lambda_{\rm eff} := \frac{R_1}{4} , \quad G_{\rm eff} := \frac{G_0}{f_R(R_1)} . \quad (11)$$

This is just GR with Λ and G_0 replaced by the *effective constants* $\Lambda_{\rm eff}$ and $G_{\rm eff}$.

Thus, we just saw that $f(R)$ gravity can reduce trivially to $GR\Lambda$ with $\Lambda \rightarrow \Lambda_{\rm eff}$ and $G_0 \rightarrow G_{\rm eff}$. A particular case is when $R_1 = 0$ (i.e. $f(0) = 0$), in which case $\Lambda_{\rm eff} = 0$. Therefore, for the problem at hand, all the BH's solutions found so far in GR in vacuum or with traceless EMT that are asymptotically flat (AF) or with a cosmological constant [i.e. asymptotically De Sitter (ADS) or Anti-De Sitter (AADS)] exist also in $f(R)$ gravity when $R = R_1 = const$:

- **Static and Spherically Symmetric (SSS) in vacuum:** Schwarzschid (AF), Schwarzschid–Λ^+ (ADS), Schwarzschid-Λ^- (AADS)

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It is then rather deceptive to find several articles in the literature reporting solutions with $R = const$ as something new or special in the context of $f(R)$ gravity, since as we just saw, such solutions are simply the same solutions found in GR but with G_0 and Λ replaced by $\Lambda_{\rm eff}$ and $G_{\rm eff}$.

Black holes and No-hair conjecture

In the context of GR, the SSS solutions, Schwarzschid and Reissner–Nordstrom (RN), and the stationary and axisymmetric ones, Kerr-Newman family (AF or with a cosmological constant) are considered as *bald* solutions:

1a) Birkhoff theorem: the AF *spherically symmetric* solution of Einstein's equations in vacuum $R_{ab} = 0$ is also *static* and therefore corresponds to the Schwarzschid solution.

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1b) Uniqueness-theorems SSS (at least valid for the AF case): the only AFSSS electro-vacuum BH solutions of Einstein's equations are the RN solution. In vacuum, the only AFSSS solution is the Schwarzschid one. The BH's solutions are therefore characterized by two parameters: M the *mass* and Q the *electric charge*.

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2) Uniqueness-theorems Stationary and Axisymmetric (SAS) (at least valid for the AF case): the only AFSAS electro-vacuum BH solutions of Einstein's equations are within the Kerr-Newman family. In vacuum, the only AFSAS solution is the Kerr solution. The BH's solutions are therefore characterized by three parameters: M the *mass* and Q the *electric charge* and J the *angular momentum.*

No hair conjecture

No-hair conjeture (J. A. Wheeler): all stationary and axisymmetric AF BH's are characterized by the three parameters M, J and Q: *BH's have no hair*.

What is hair ? Loosely speaking...

Def: any quantity other than M, Q, J (e.g. more multipoles) required to characterize an AF black hole

The original no-hair conjecture turns to be false:

Exemples of Hair in AFSSS BH's: (zero charges) scalar-hair (Einstein-scalar field system with a suitable potential), colored black holes (Einstein-Yang-Mills). Nevertheless, the hair turns to be unstable.

In principle one could generalize such hairy solutions to stationary and axisymmetric spacetimes.

No-hair conjecture (revisited): there is no stable hair in AFSSS or AFSAS BH's.

No hair theorems in GR

These theorems have been proved (mostly) under the AF and stationary or SSS assumptions. They provide the necessary and sufficient conditions for the absence of BH's solutions endowed with non-trivial matter fields. Here we remind the no-hair theorem that will be relevant for $f(R)$ gravity:

No scalar-hair theorem (Sudarsky 1995; Bekenstein 1995)*: Assume AFSSS spacetime and the Einstein-ϕ system (i.e. Einstein equations with an EMT of a real scalar field minimally coupled to gravity). If the potential $\mathscr{U}(\phi)$ associated with this system satisfies the condition $\mathscr{U}(\phi) \geq 0$, then the only regular BH is the Schwarzschid BH. That is, the only solution admited for $\phi(r)$ is the trivial one $\phi(r) = \phi_0 = const$ such that $\mathscr{U}(\phi_0) = 0 \to AF$ (i.e. no effective cosmological constant, as opposed to $\kappa\Lambda_{\rm eff} = \mathscr{U}(\phi_0) \neq 0$).

*Remark 1: the theorem applies also for multiple real scalar-fields.

Remark 2: BH's with scalar-hair have been found when $\mathscr{U}(\phi)$ does not satisfy the condition $\mathscr{U}(\phi) \geq 0$ (e.g. Nucamendi & MS 2003; Anabalon & Oliva 2012)

(Quest for) Hairy Black holes in (vacuum) $f(R)$ gravity

As mentioned before, when one consider solutions $R = const$ then $f(R)$ gravity reduces to $GR + \Lambda_{\rm eff} + G_{\rm eff}$, and so one finds the same BH's as in GR, in particular in vacuum. What about BH's with non trivial R ? That is, BH's with a non trivial Ricci scalar $R(r)$ interpolating from the BH's horizon at $r = r_h$ and spatial infinity $r = \infty$? In what follows we shall show *analytical* (i.e. application of the NHT's) and numerical evidence about the absence of such non-trivial SSSAFBH solutions. To do so, we focus on SSS spacetimes in vacuum.

But prior to that, let us just mention that in the context of $f(R)$ gravity the following issues are open questions (as far as we are aware):

- **1)** In general: No analogue of rigorous Birkhoff theorem (we can view $f(R)$ gravity as GR endowed with an exotic effective EMT).
- **2)** In general: No uniqueness BH's theorems.
- **3)** In general: absence of no-hair theorems.

One has to analyze these issues in a case-by-case basis, i.e. for each specific $f(R)$ model.

It turns however that when the $f(R)$ model satisfies the condition $0 < f_{RR} < \infty$ ($f_{RR} < 0$ leads to the Dolgov-Kawazaki instability –negative mass modes–) and $f_R > 0$ (i.e. $G_{\rm eff} = G_0/f_R > 0$) one can map the theory into a Scalar-Tensor-Theory in the *Einstein frame*. This is done by a sequences of steps, where the most crucial one is a conformal transformation (see the next slide).

The important point is that in vacuum, the $f(R)$ theory in the EF has exactly the same form as the Einstein-ϕ system, where the scalar-field ϕ turns to be coupled minimially to the curvature, but with an "exotic" scalar-field potential $\mathscr{U}(\phi)$.

Therefore one can use the NHT's to check if scalar-hair is ruled out or not. For instance, if $\mathscr{U}(\phi) \geq 0$ then the NHT's rule out the existence of a non-trivial $\phi(r)$ and thus a non trivial $R(r)$.

$$\chi = f_R \tag{12}$$

$$\phi = \sqrt{\frac{3}{2\kappa}}\ln\chi \;, \tag{13}$$

$$\tilde{g}_{ab} = \chi\, g_{ab} = e^{\sqrt{\frac{2\kappa}{3}}\phi}\, g_{ab} \;, \tag{14}$$

$$V(\chi) = \frac{1}{2\kappa\chi^2}\Big[R(\chi)\chi - f(R(\chi))\Big] \;, \tag{15}$$

$$\mathscr{U}(\phi) := V(\chi[\phi]) \tag{16}$$

In terms of these new variables the gravitational action

$$I_{\rm JF}[g_{ab}] = \int \frac{f(R)}{2\kappa}\sqrt{-g}\, d^4x \;, \tag{17}$$

takes the form

$$I_{\rm EF} = \int d^4x\, \sqrt{-\tilde{g}}\Big[\frac{1}{2\kappa}\tilde{R} - \frac{1}{2}\tilde{g}^{ab}(\tilde{\nabla}_a\phi)(\tilde{\nabla}_b\phi) - \mathscr{U}(\phi)\Big], \tag{18}$$

where all the quantities with a *tilde* are defined with respect to the conformal metric $\tilde{g}_{ab}$. This is just the Einstein-Hilbert action with a scalar field coupled minimally to gravity.

Unlike the JF action, a kinetic term appears due to the conformal transformation.

The field equations obtained from the EF action are simply those of the Einstein-ϕ system:

$$\tilde{G}_{ab} = \kappa T^{\phi}_{ab} \ , \tag{19}$$

$$T^{\phi}_{ab} = (\tilde{\nabla}_a\phi)(\tilde{\nabla}_b\phi) - \tilde{g}_{ab}\left[\frac{1}{2}\tilde{g}^{cd}(\tilde{\nabla}_c\phi)(\tilde{\nabla}_d\phi) + \mathscr{U}(\phi)\right] \ , \tag{20}$$

$$\Box\phi = \frac{d\mathscr{U}}{d\phi} \ . \tag{21}$$

Let us now recall the no-hair theorems mentioned previously

No scalar-hair theorem (Sudarsky 1995; Bekenstein 1995)*: Assume SSSAF spacetime and the Einstein-ϕ system. If the potential $\mathscr{U}(\phi)$ satisfies the condition $\mathscr{U}(\phi) \geq 0$, then the only regular BH is the Schwarzschid BH. That is, the only solution admited for $\phi(r)$ is the trivial one $\phi(r) = \phi_0 = const$ such that $\mathscr{U}(\phi_0) = 0 \rightarrow AF$ (i.e. no effective cosmological constant, as opposed to $\kappa\Lambda_{\rm eff} = \mathscr{U}(\phi_0) \neq 0$).

So given a $f(R)$ model it is sufficient to compute $\mathscr{U}(\phi)$:

1) If $\mathscr{U}(\phi) \geq 0$ then there is no scalar hair $\longrightarrow$ only $R(r) = 0$ (trivial) BH's solution: Schwarzschild.

2) If $\mathscr{U}(\phi)$ has negative branches or is not well defined (multivalued) (if $f_{RR} > 0$ fails) $\longrightarrow$ no-hair theorems do not apply $\longrightarrow$ numerical solutions.

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Which $f(R)$ model ?

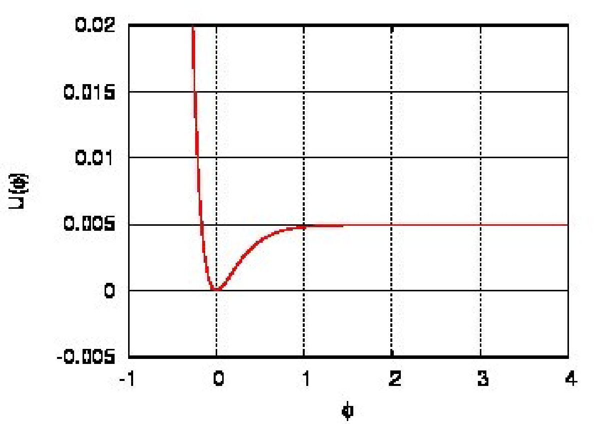
We explore several models used in the past and proposed to solve several cosmological problems (dark energy, inflation, etc.)

Model	Potential in EF	Properties	No-hair theorem
$f(R)=\lambda\left(\frac{R}{R_n}\right)^n$	$\mathscr{U}(\phi)=\frac{(n-1)}{2\kappa n}\left(\frac{R_n}{\lambda n}\right)^{\frac{1}{n-1}} e^{\left(\frac{2-n}{n-1}\right)\sqrt{\frac{2\kappa}{3}}\phi}$	$\mathscr{U}(\phi)>0\ , R_n>0 , \lambda>0 , n>0$	✓
$f(R)=R+c_2\left(\frac{R}{R_I}\right)^2$	$\mathscr{U}(\phi)=\frac{R_I}{8c_2\kappa}\left(1-e^{-\sqrt{\frac{2\kappa}{3}}\phi}\right)^2$	$\mathscr{U}(\phi)\geq 0\ , c_2>0$ $\mathscr{U}(\phi)\leq 0\ , c_2<0$	✓ ✗
$f(R)=R-\alpha_1 R_* \ln\left(1+\frac{R}{R_*}\right)$	$\mathscr{U}(\phi)=\frac{R_*}{2\kappa}e^{-2\sqrt{\frac{2\kappa}{3}}\phi}\left[\alpha_1\ln\left(\frac{\alpha_1}{1-e^{\sqrt{\frac{2\kappa}{3}}\phi}}\right)-e^{\sqrt{\frac{2\kappa}{3}}\phi}+1-\alpha_1\right]$	$\mathscr{U}(\phi)>0\ , 1\leq\alpha_1 ,\ R_*>0$ $\mathscr{U}(\phi)\geq 0\ ,\ 0<\alpha_1<1\ ,\ R_*>0$	✓ ✓
$f(R)=R-R_e\lambda_e\left(1-e^{-\frac{R}{R_e}}\right)$	$\mathscr{U}(\phi)=\frac{R_e}{2\kappa}e^{-2\sqrt{\frac{2\kappa}{3}}\phi}\left[\left(e^{\sqrt{\frac{2\kappa}{3}}\phi}-1\right)\times\ln\left(\frac{\lambda_e}{1-e^{\sqrt{\frac{2\kappa}{3}}\phi}}\right)+\lambda_e+e^{\sqrt{\frac{2\kappa}{3}}\phi}-1\right]$	$\mathscr{U}(\phi)>0\ , 1\leq\lambda_e ,\ R_e>0$ $\mathscr{U}(\phi)\geq 0\ , 0<\lambda_e<1 ,\ R_e>0$ $\mathscr{U}(\phi)\leq 0\ , \lambda_e<0 ,\ R_e>0$	✓ ✓ ✗
$f(R)=R+\lambda_S R_S\left[\left(1+\frac{R^2}{R_S^2}\right)^{-q}-1\right]$	(ill defined: multivalued)	–	–
$f(R)=R-R_{\rm HS}\frac{c_1\left(\frac{R}{R_{\rm HS}}\right)^n}{c_2\left(\frac{R}{R_{\rm HS}}\right)^n+1}$	(ill defined: multivalued)	–	–

TABLE I: $f(R)$ models and their corresponding scalar-field potentials in the Einstein frame. The last column indicates if the no-hair theorem aplies (✓) or not (✗). The potentials that are striclty positive definite $\mathscr{U}(\phi)>0$, do not even admit Schwarzschild BH's with $R(r)=0$. When the no-hair theorems cannot be applied, in particular when the EFSTT approach is not well defined, we perform a numerical analysis in order to conclude the existence or absence of geometric hair in AFSSS black holes.

Model 2:

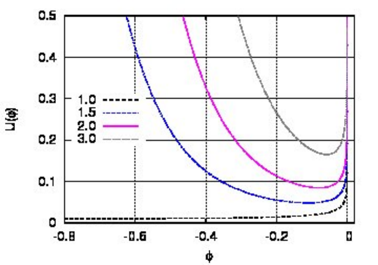
$f(R) = R + c_2 R_I (R/R_I)^2$ (Starobinsky 1980)



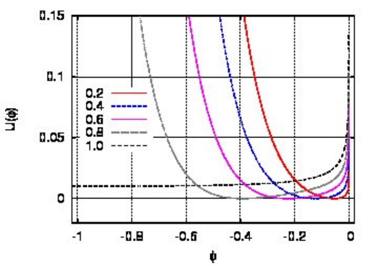
Model 3:

$$f(R) = R - \alpha_1 R_* \ln\left(1 + \frac{R}{R_*}\right) \text{ (Miranda } \textit{et al } 2009)$$

$1 \leq \alpha_1$



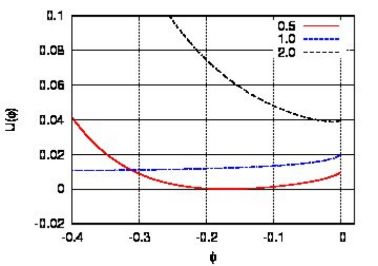
$0 < \alpha_1 \leq 1$



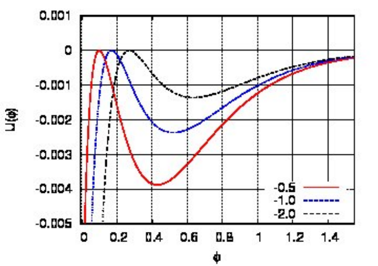
Model 4:

$f(R) = R - R_e\lambda_e\left(1 - e^{-\frac{R}{R_e}}\right)$ (Cognola *et al* 2008; Linder 2009; Yang *et al* 2010; Bamba *et al* 2010)

$0 < \lambda_e$



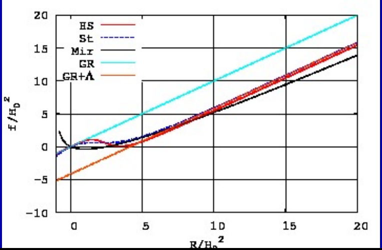
$\lambda_e < 0$

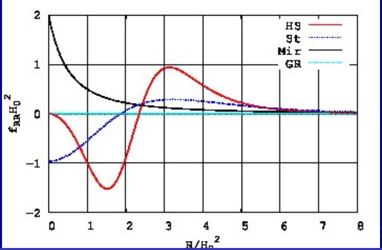


Model 5:

$f(R) = R + \lambda_S R_S \left[\left(1 + \frac{R^2}{R_S^2}\right)^{-q} - 1 \right]$ where q, λ_S, R_S are parameters of the model (Starobinsky 2007). R_S sets the scale.

Model 6: $f(R) = R - R_{\rm HS} \frac{c_1 \left(\frac{R}{R_{\rm HS}}\right)^n}{c_2 \left(\frac{R}{R_{\rm HS}}\right)^n + 1}$, where c_1 and c_2 are two dimensionless parameters, and like in previous models, $R_{\rm HS}$ simply sets the scale.





Models 1-4 have $f_{RR} > 0$ in the domain where $f(R)$ itself is defined. However, Models 5-6 do not. Therefore, the STT approach cannot be applied in general since $\chi = f_R$ cannot be inverted in all the domain where $f(R)$ is defined. It coud be done so "piecewise". The potential $\mathscr{U}(\phi)$ cannot be given in closed form, however one can draw the potential using the following parametric representation for the models 5 and 6

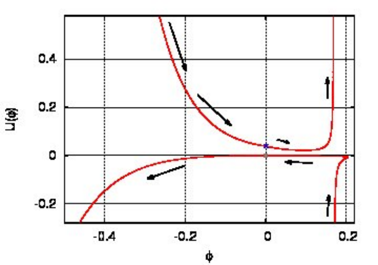
$$\chi(R) = f_R = 1 - \frac{2\lambda_s q(R/R_s)}{\left[1 + (R/R_S)^2\right]^{1+q}} \quad \text{(Model 5)} \tag{22}$$

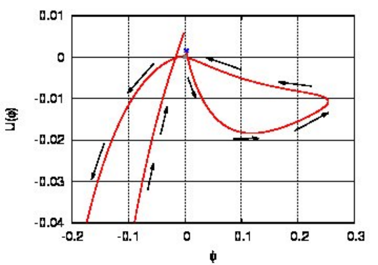
$$\chi(R) = f_R = 1 - \frac{nc_1(R/R_{\rm HS})^{n-1}}{\left[1 + c_2(R/R_{\rm HS})^n\right]^2} \quad \text{(Model 6)} \tag{23}$$

$$\phi(R) = \sqrt{\frac{3}{2\kappa}\ln\chi(R)}\ , \tag{24}$$

$$\mathscr{U}(\phi(R)) := V(\chi[\phi(R)])\ , \tag{25}$$

Models 5,6





The potentials are multivalued. Even if one can make sense to them for some branches one should clearly specify for which branches the potentials make sense. In any case, the potentials $\mathscr{U}(\phi)$ does not satisfy the condition $\mathscr{U}(\phi) \geq 0$ for the No-hair theorems to apply. Thus we require a numerical analisys in the original variables where everything is a priori well defined.

AFSSS Black holes in $f(R)$ gravity

So let us consider a SSS spacetime:

$$ds^2 = -\left(1 - \frac{2M(r)}{r}\right) e^{2\delta(r)} dt^2 + \left(1 - \frac{2M(r)}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad . \quad (26)$$

After lenghty calculations the $f(R)$ field equations yield [see L. Jaime, L. Patiño, M. Salgado, PRD **83**, 024039 (2011)] (Remark: in GR, with $\Lambda = 0$, $f(R) = R$, $f_R = 1$, $f_{RR} = 0$, $f_{RRR} = 0$. In vacuum $R = 0$. Cf. the terms in green below)

$$R'' = \frac{1}{3f_{RR}}\left[\frac{(2f - Rf_R)r}{r - 2M} - 3f_{RRR}R'^2\right] + \left[\frac{2(rM' - M)}{(r-2M)r} - \delta' - \frac{2}{r}\right]R' \quad (27)$$

$$M' = \frac{M}{r} + \frac{1}{2(2f_R + rR'f_{RR})}\left\{-\frac{4f_R M}{r} + \frac{r^2}{3}(Rf_R + f)\right.$$
$$\left. + \frac{rR'f_{RR}}{f_R}\left[\frac{r^2}{3}(2Rf_R - f) - \frac{4M}{r}f_R + 2rR'f_{RR}\left(1 - \frac{2M}{r}\right)\right]\right\}, \quad (28)$$

$$\left(1 - \frac{2M}{r}\right)\delta' = \frac{1}{2(2f_R + rR'f_{RR})r}\left\{\frac{2r^2}{3}(2f - Rf_R) + \frac{rR'f_{RR}}{f_R}\left[\frac{r^2}{3}(2Rf_R - f) - 2f_R\right]\right.$$
$$\left. - 4(r-2M)R'f_{RR} + \frac{2(f_R + rR'f_{RR})(r-2M)R'f_{RR}}{f_R}\right\}. \quad (29)$$

Some remarks:

1) As indicated in green, when $f(R) = R$ we recover the Eqs. of GR whose unique regular AF solution is the Schwarzschild solution. If $f(R) = R - 2\Lambda$ we recover the Eqs. of GR+Λ whose unique regular solution is the Schwarzschild-(A)DS solution.

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3) Solve the above ODE's from the horizon $r = r_h$ (defined by the Killing horizon $g_{tt}(r_h) = 0$) to spatial infinity $r = \infty$. As mentioned we focus here only for *AF* spacetimes ($g_{tt} \rightarrow -1$, $g_{rr} \rightarrow 1$, $\partial_r g_{tt} \rightarrow\sim \mathcal{O}(1/r^2)$, $\partial_r g_{rr} \rightarrow\sim \mathcal{O}(1/r^2)$. The mass function $M(r)$ has to converge to the Komar (ADM) mass at $r = \infty$.

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4) Equations require BC $\to$ *regularity conditions* at r_h and the AF condtion at $r = \infty$. The former are provided below. The AF condition requires $R \to 0$ as $r \to \infty$. This is enforced by solving the EOM for R using a *shooting method.*

Regularity conditions at the horizon: The field equations must be satisfied at r_h "smoothly". We expand the metric functions $\delta(r)$, $M(r)$ and also $R(r)$ as

$$F(r) = F(r_h) + (r - r_h)F'(r_h) + \frac{1}{2}(r - r_h)F''(r_h) + \frac{1}{6}(r - r_h)^3 F'''(r_h) + \mathcal{O}(r - r_h)^4 \quad (30)$$

where $F(r)$ stands for $M(r)$, $R(r)$, $\delta(r)$. When replacing these expansions in the equations and demand that the derivatives of these variables are finite at the horizon one obtains after long but straightforward algebra the following regularity condition:

$$R'|_{r=r_h} = \left. \frac{2r\left(Rf_R - 2f\right)f_R}{\left[r^2(2Rf_R - f) - 6f_R\right]f_{RR}} \right|_{r=r_h} , \quad (31)$$

$$M'|_{r=r_H} = \left. \frac{r^2\left(2Rf_R - f\right)}{12f_R} \right|_{r=r_h} , \quad M_h = \frac{r_h}{2} \; , \delta(r_h) = 0 \; , \quad (32)$$

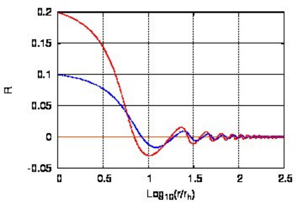
where as stressed, all the quantities in the above equation are to be evaluated at the horizon $r = r_h$. The expressions for higher derivatives at $r = r_h$ are very long and not very enlightening (see arXiv: 1509.01664 for the details).

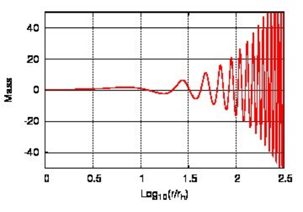
We constructed a FORTRAN code to solve numerically the previous set of ODE's submitted to the regularity conditions we just showed in order to find non-trivial hairy solutions. We perfomed several tests to assess the accuracy of our results, like comparing with some exact solutions found in the literature (I have no time to show such solutions: they are asymptotically flat except for a deficit angle: $R(r) = 1/r^2$).

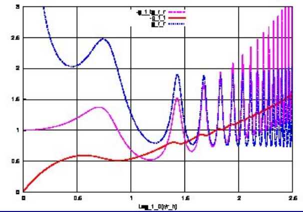
Warning: for the BH solution to be genuinely AF the mass function $M(r)$ must converge to a definite value at $r = \infty$. This is the Komar or ADM mass. For spacetimes that have a deficit angle $M(r) \sim r$ asymptotically, and those that have an effective cosmological constant $M(r) \sim r^3$.

Numerical solutions

Model 5: Starobinsky $q = 2$

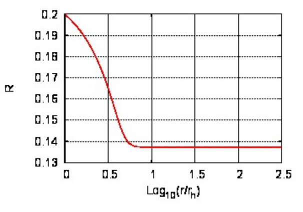


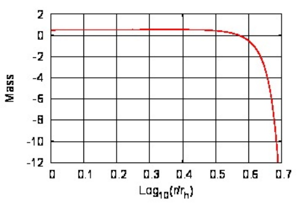


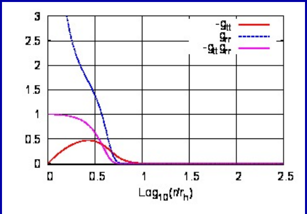


Numerical solutions

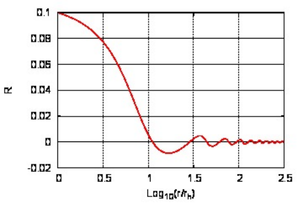
Model 5: Starobinsky $q = 4$

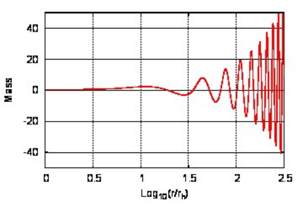


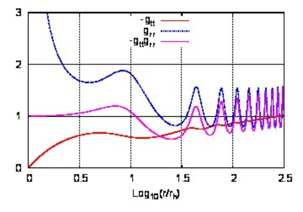




Model 6: Hu–Sawicky $n = 1$







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- When no-hair theorems do not apply, numerical analysis is required. This indicates that hair is absent. That is, the only AFSSSBH regular solution seems to be the Schwarzschild solution.

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- No analogue of a rigorous Birkhoff theorem exists. No uniquess theorems of AFSSS BH's exist.
- Using STT approach No-Hair theorems can be applied for some $f(R)$ models.
- When no-hair theorems do not apply, numerical analysis is required. This indicates that hair is absent. That is, the only AFSSSBH regular solution seems to be the Schwarzschild solution.
- Work in progress: analysis on existence or absence of hair in SSS solutions in AADS and ADS backgrounds.

The staticity condition implies that the EOM are invariant to the transformation $\delta(r) \to \delta(r) + const$. This means that we are free to fix $\delta(r_h)$ by adjusting the constant. So we choose the simple value $\delta(r_h) = 0$.
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Moreover, the regularity conditions for $R''(r)$, $\delta'(r)$ are

$$\begin{aligned} \delta'\Big|_{r_h} &= 4r^3\Big\{r^2 f f_{RR}\Big[f\Big(10f - 13Rf_R\Big) + \frac{6f_R}{r^2}\Big(2f - Rf_R\Big)\Big] + r^2 f f_R^2\Big[10f - R\Big(13f_R - 4Rf_{RR}\Big)\Big] \\ &+ 2f_R^3\Big[6f + Rf_R\Big(2r^2R - 3\Big)\Big]\Big\}\Big\{f_{RR}\Big[2f_R\Big(3 - r^2R\Big) + r^2 f\Big]\Big[\Big(-41r^2 f + 2f_R\Big(13r^2R - 63\Big)\Big)r^2 Rf_R \\ &+ 2r^2\Big(7r^2 f^2 + 18f_R\Big(\frac{4f_R}{r^2} + 3f\Big)\Big)\Big]\Big\}^{-1}\Big|_{r_h} , \end{aligned} \tag{33}$$

$$\begin{aligned} R''\Big|_{r_h} &= -4r^2 f_R\Big(-Rf_R + 2f\Big)\Big\{r^2 f^2\Big[-12r^2 f f_{RR}^2 + 3r^2 f_R f_{RR}\Big(-4f_R + f_{RR}\Big(9R - \frac{20}{r^2}\Big)\Big) \\ &+ 4r^2 f_R f_{RRR}\Big(7f - 6f_R\Big(4R - \frac{9}{r^2}\Big)\Big)\Big] + f_R^2\Big[r^2 Rf_R f_{RRR}\Big(2f_R\Big(63R - \frac{72}{r^2} - 13r^2R^2\Big) \\ &- 3f\Big(120 - 31r^2R\Big)\Big) + r^2 f_R f_{RR}\Big(3f_R\Big(22R - 5r^2R^2 - \frac{24}{r^2}\Big) - 3f\Big(20 - 9r^2R\Big)\Big) \\ &+ 288 f f_R f_{RRR} + 3r^2 f f_{RR}^2\Big(22R - \frac{24}{r^2} - 5r^2R^2\Big)\Big]\Big\} \\ &\times \Big\{f_{RR}^3\Big[14r^2 f + f_R\Big(24 - 13r^2R\Big)\Big]\Big[r^2 f + 2f_R\Big(3 - r^2R\Big)\Big]^3\Big\}^{-1}\Big|_{r_h} . \end{aligned} \tag{34}$$

Some exact solutions (a small detour)

$$f(R) = 2a\sqrt{R-\alpha} = 2a^2\sqrt{\left(\frac{R}{a^2}\right) - \left(\frac{\alpha}{a^2}\right)} \ , \tag{35}$$

where $a > 0$ is a parameter with units $[\text{distance}]^{-1}$, and α is another parameter of the model which is related to an effective cosmological constant as we show below. In the SSS scenario the metric

$$ds^2 = -\frac{1}{2}\left(1 - \frac{\alpha r^2}{6} + \frac{2Q}{r^2}\right)dt^2 + \frac{dr^2}{\frac{1}{2}\left(1 - \frac{\alpha r^2}{6} + \frac{2Q}{r^2}\right)} + r^2 d\Omega^2 \ , \tag{36}$$

with the mass function, Ricci scalar, and $\delta(r)$ given, respectively, by

$$M(r) = \frac{r}{4} + \frac{\alpha r^3}{24} - \frac{Q}{2r} \ , \tag{37}$$

$$R(r) = \alpha + \frac{1}{r^2} \ , \tag{38}$$

$$\delta(r) \equiv 0 \ , \tag{39}$$

solve Eqs. (27)–(29) exactly, as one can verify by straightforward calculations. Here, Q is an integration constant. When $\alpha = 0$ this solution was part of a more general class of solutions associated with the model $f(R) = kR^n$ (Clifton 2006; Nzioki et al. 2010)

However, those authors did not analyze in detail the exact solution.

Notice that in the absence of a cosmological constant $\alpha = 0$ the solution behaves asymptotically as $g_{rr} \to 1/2$ and $g_{tt} \to 1/2$ and $R \to 0$.

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$$ds^2 = -\left(1 - \Delta - \frac{2M}{r} + \frac{q^2}{r^2}\right)dt^2 + \frac{dr^2}{\left(1 - \Delta - \frac{2M}{r} + \frac{q^2}{r^2}\right)} + r^2 d\Omega^2 \ , \qquad (40)$$

with $\Delta = 1/2$, $M = 0$ and $Q = \pm q^2$.
After a redefinition of coordinates $r = (1-\Delta)^{1/2}\tilde{r}$, $t = (1-\Delta)^{-1/2}\tilde{t}$, $M = M_{ADM\Delta}(1-\Delta)^{3/2}$ the metric takes the standard form

$$ds^2 = -\left(1 - \frac{2M_{ADM\Delta}}{\tilde{r}} + \frac{\mathcal{Q}^2}{\tilde{r}^2}\right)d\tilde{t}^2 + \frac{d\tilde{r}^2}{\left(1 - \frac{2M_{ADM\Delta}}{\tilde{r}} + \frac{\mathcal{Q}^2}{\tilde{r}^2}\right)} + (1-\Delta)\tilde{r}^2 d\Omega^2 \ , \quad (41)$$

where $q = \mathcal{Q}(1-\Delta)$. The quantity $\mathcal{Q}$ is presumably the actual charge when $\Delta \neq 0$. Again, taking $M_{ADM\Delta} \equiv 0$, $\Delta = 1/2$, we recover the metric (36) written in the standard form but now with $Q \neq 0$ given by $Q = \pm q^2 = \pm \mathcal{Q}^2/4$.

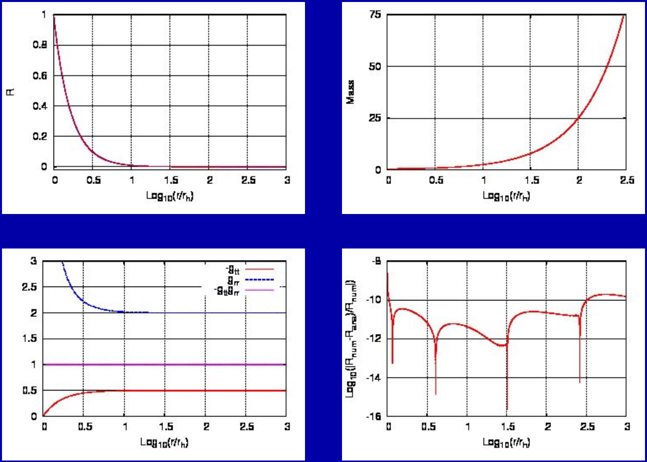
Q: So what ?

The important point for doing all this is the last term of the metric: $ds^2_{\rm ang} = (1-\Delta)\tilde{r}^2 d\Omega^2$. The metric has a solid deficit angle Δ !

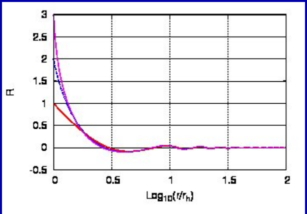
In summary, the exact solution corresponds to a spacetime containing a Black Hole with zero mass and with a solid angle deficit.
Spacetimes with a deficit angle has Ricci scalar $R(r) = 2\Delta/r^2$. Therefore the fact that $R(r) \neq const$ does not mean that the solution has hair !

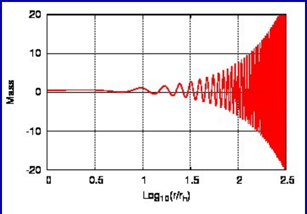
See Nucamendi & Sudarsky 2000 for a discussion about the meaning of zero mass BH's in this kind of spacetimes.
The quest for hair is very subtle and one has to be very careful with the analysis of the asymptotics to claim that there is hair.

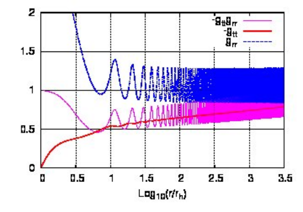
Model: $f(R) = 2a\sqrt{R-\alpha}$. Exact and numerical solution:



Model 4: Exponential $f(R) = R - R_e\lambda_e(1 - e^{-\frac{R}{R_e}})$ with $\lambda_e < 0$







$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \quad , \qquad (42)$$

where $k = \pm 1, 0$. When obtaining numerical solutions we shall focus only on the "flat" case $k = 0$.

$$H^2 + \frac{k}{a^2} + \frac{1}{f_R}\left[f_{RR}H\dot{R} - \frac{1}{6}\left(f_R R - f\right)\right] = \frac{\kappa\rho}{3f_R} \quad , \qquad (43)$$

$$\ddot{a}/a = \dot{H} + H^2 = \underbrace{\frac{1}{f_R}\left(f_{RR}H\dot{R} + \frac{f}{6} - \frac{\kappa\rho}{3}\right)}_{R_1/12=\Lambda_{\rm eff}/3 \;\; when \;\; \rho\to 0 \;\; and \;\; if \;\; R\to R_1} \quad , \qquad (44)$$

where $H = \dot{a}/a$ is the Hubble expansion. From Eq. (4) we find

$$\ddot{R} = -3H\dot{R} - \frac{1}{3f_{RR}}\left[3f_{RRR}(\dot{R})^2 + \underbrace{2f - f_R R}_{V'(R)} - \kappa(\rho - 3p)\right] \quad . \qquad (45)$$

If $R(t)$ reaches R_1 of the potential $V(R)$ (with vanishing $\dot{R}$,and $\ddot{R}$), in the far future where the matter contributions $\rho, p \ll \rho_{\rm crit}$ and $R \approx R_1$ today, then $\dot{H} + H^2 = \ddot{a}/a \approx R_1/12 = \Lambda_{\rm eff}/3 > 0$ if $\Lambda_{\rm eff} > 0$. Thus $\ddot{a} > 0 \to$ Accelerated expansion !!. This what happens precisely when solving the full equations numerically taking into account all the matter terms.

Now, the expresion for the Ricci scalar is given by

$$R = 6\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right) \quad . \tag{46}$$

Note that by using Eqs. (43) and (44) in Eq. (46) we obtain an identity $R \equiv R$, which shows the consistency of the equations (c.f. the SSS case) !
T_{ab} of matter is a mixture of three kinds of perfect fluids: baryons, radiation and dark matter, in a epoch where they don't interact with each other except gravitationally. Then for each matter component the EMT conserves separately and $\nabla_a T_i^{ab} = 0$ ($i = 1, 3 \rightarrow$ baryons, radiation, DM) leads to

$$\dot{\rho}_i = -3H\left(\rho_i + p_i\right) \quad . \tag{47}$$

The total energy-density is $\rho = \sum_i \rho_i = -T^t{}_t$ and since $p_{\rm bar,DM} = 0$, and $p_{\rm rad} = \rho_{\rm rad}/3$ then $T = T_{\rm bar} + T_{\rm DM} = -(\rho_{\rm bar} + \rho_{\rm DM})$. Then Eq. (47) integrates

$$\rho = \frac{\rho^0_{\rm bar} + \rho^0_{\rm DM}}{(a/a_0)^3} + \frac{\rho^0_{\rm rad}}{(a/a_0)^4} \quad , \tag{48}$$

where the knotted densities are the densities today. Here $a_0 = a(t_0)$, t_0 is present cosmic time. The differential equations will depend explicitly on $a(t)$ via the matter terms.

In the ΛCDM model the equation of state $\omega_\Lambda = p_\Lambda/\rho_\Lambda = -1$. We shall define an EOS for the modified gravity contribution given by (for $f(R) \neq R$)

$$\omega_X = \frac{p_X}{\rho_X} \quad , \tag{49}$$

where ρ_X is defined from the modified Friedmann equation, so that it reads $H^2 = \frac{\kappa}{3}\left(\rho + \rho_X\right) = \frac{\kappa \rho_{\rm tot}}{3}$, which leads to

$$\rho_X = \frac{1}{\kappa f_R}\left\{\frac{1}{2}\left(f_R R - f\right) - 3 f_{RR} H \dot{R} + \kappa\rho\left(1 - f_R\right)\right\} , \tag{50}$$

In a similar way we define ρ_X, so that the dynamic equation for H reads

$$\dot{H} + H^2 = -\frac{\kappa}{6}\left\{\rho + \rho_X + 3\left(p_{\rm rad} + p_X\right)\right\} = -\frac{\kappa \rho_{\rm tot}}{6}\left\{1 + 3\omega_{\rm tot}\right\} \quad , \tag{51}$$

where $\omega_{\rm tot} = p_{\rm tot}/\rho_{\rm tot}$. From this latter, we obtain

$$p_X = -\frac{1}{3\kappa f_R}\left[\frac{1}{2}\left(f_R R + f\right) + 3 f_{RR} H \dot{R} - \kappa\left(\rho - 3 p_{\rm rad} f_R\right)\right] \tag{52}$$

Among the infinite a priori possible choices of $f(R)$ (restricted by $f_R > 0$ so as to $G_{\rm eff} = G_0/f_R > 0$ and $f_{RR} > 0$, stable perturbations around a background), how to choose ?

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$-$ Simplicity $\rightarrow f(R) = R - 2\Lambda$. But we don't want this. We want something with $\omega_X(t)$ such that today $\omega_x \approx -1$.

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– Is there any new physical principle that single out an $f(R)$ different from $f_{GR}(R)$, that match all the tested gravitational observations and yet provide new and "unexpected" predictions ? Ans. Maybe.

Given a specific $f(R)$, we integrate the differential equations forward from past to future with suitable "initial conditions". We have considered three specific $f(R)$ models which have become very popular in the literature

- Miranda *et. al.* model (PRL **102**, 221101, 2009)

$$f(R)_{\rm MJW} = R - \beta R_* \ln\left(1 + \frac{R}{R_*}\right) \quad . \tag{53}$$

We used $\beta = 2$ and $R_* = H_0^2$.

- Starobinsky model (JETP Lett. **86**, 157 2007)

$$f(R)_{\rm St} = R + \lambda R_S \left[\left(1 + \frac{R^2}{R_S^2}\right)^{-q} - 1\right] \quad . \tag{54}$$

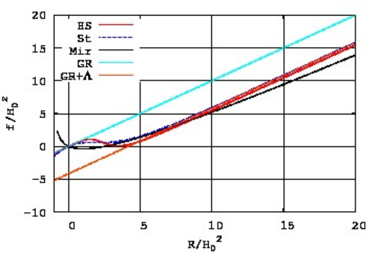
We take $q = 2$ and $\lambda = 1$, $R_S \approx 4.17 H_0^2$.

- Hu & Sawicky model (PRD **76**, 064004, 2007)

$$f(R)_{HS} = R - m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1} \quad . \tag{55}$$

We take $n = 4$, $m^2 \approx 0.24 H_0^2$, $c_1 \approx 1.25 \times 10^{-3}$ and $c_2 \approx 6.56 \times 10^{-5}$.

$f(R)$ Models


HS
St
Mir
GR
GR+Λ
20
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−5
−10
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5
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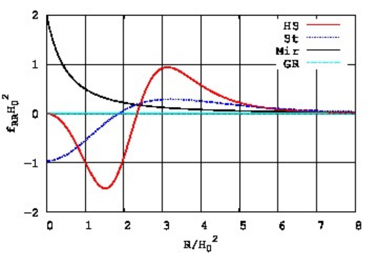
20
f/H_0^2
R/H_0^2


HS
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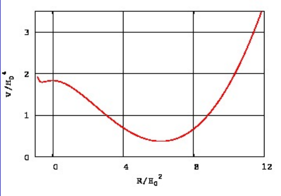
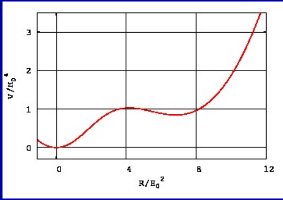
GR
2
1
0
−1
−2
0
1
2
3
4
5
6
7
8
$f_{RR}H_0^2$
R/H_0^2

Potentials $V(R) = -Rf(R)/3 + \int^R f(x)dx$ such that $V'(R) = \frac{1}{3}(2f - Rf_R)$. At the extrema of $V(R)$ (notably at the global minimum) the de Sitter "point" is reached where the models behave as a GR plus $\Lambda_{\rm eff} = R_1/4$, where $V'(R_1) = 0$. The specific cosmological models interpolate between a large R (at early time) and near the nontrivial minimum $R_1 \neq 0$ at present time.

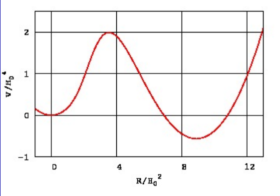
$f(R)_{MJW}$



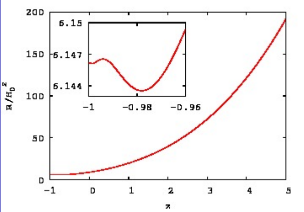
$f(R)_{St}$



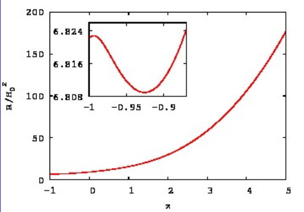
$f(R)_{HS}$



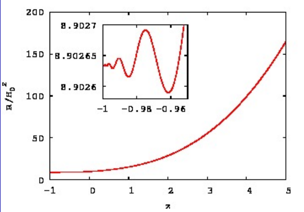
$f(R)_{MJW}$



$f(R)_{St}$

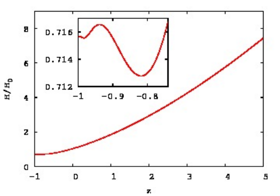


$f(R)_{HS}$

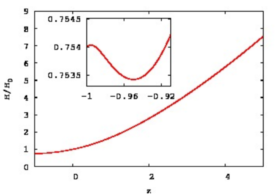


Plot $\frac{H}{H_0}$ vs z

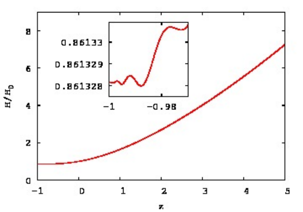
$f(R)_{MJW}$



$f(R)_{St}$



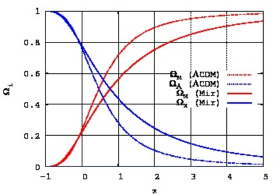
$f(R)_{HS}$



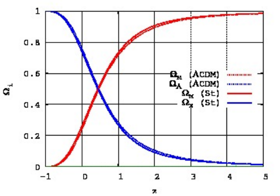
Plot Ω_i vs z

$\Omega_i = \kappa\rho_i/(3H^2)$, $i = \mathrm{matt}, \mathrm{rad}, X$; $\mathrm{matt} = baryons + DM$.

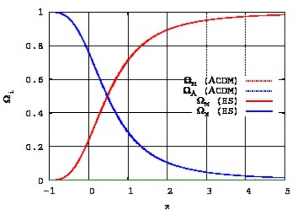
$f(R)_{MJW}$



$f(R)_{St}$

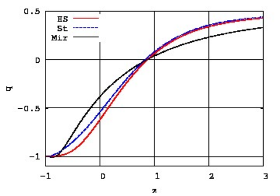


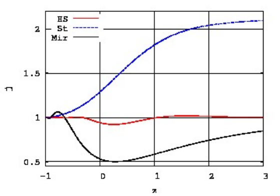
$f(R)_{HS}$



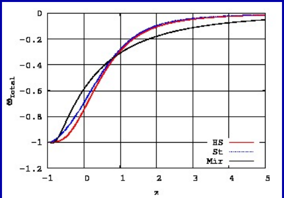
Deceleration parameter: $q := -\frac{\ddot{a}}{aH^2} = -\frac{H^2+\dot{H}}{H^2} = 1 - \frac{R}{6H^2} = \frac{1}{2}\left(1 + 3\omega_{tot}\right)$,

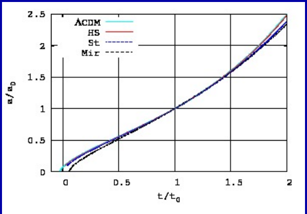
$\omega_{tot} = -\frac{1}{3}\left[\frac{\frac{1}{2}\left(f_R R + f\right) + 3f_{RR}H\dot{R} - \kappa\rho}{\frac{1}{2}\left(f_R R - f\right) - 3f_{RR}H\dot{R} + \kappa\rho}\right]$. Jerk: $j := \frac{\dddot{a}}{aH^3} = \frac{\dot{R}}{6H^3} - \frac{\dot{H}}{H^2} + 1 = \frac{\dot{R}}{6H^3} + q + 2$



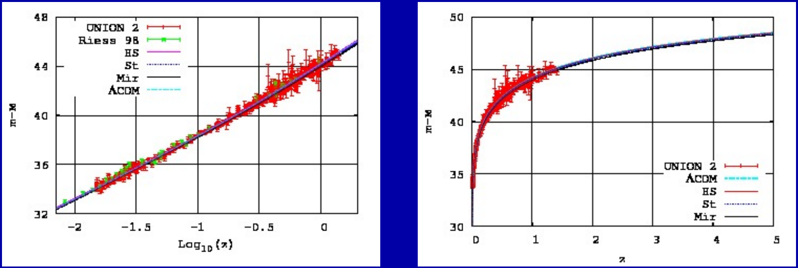


The age of the Universe: $\sim t_0 = H_0^{-1} \approx 9.78h^{-1} \times 10^9\,\mathrm{y} \sim 13.97 \times 10^9\,\mathrm{y}$ (with $h = 0.7$)

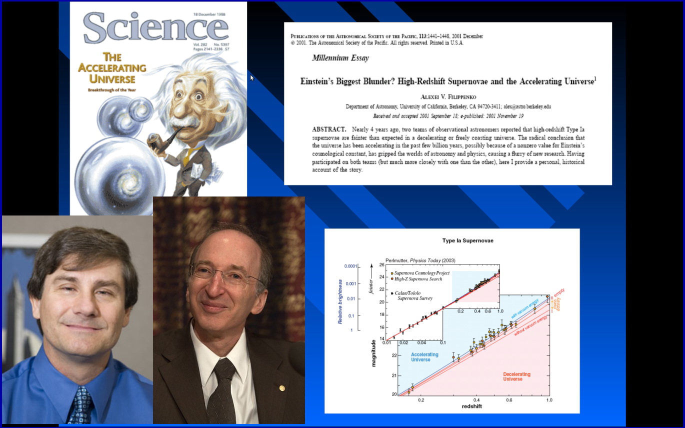




Luminosity distance and SNIa data confrontation ($k=0$): $d_L^{\rm flat} = \frac{\zeta(\bar{a})}{\bar{a}}$, where $\zeta = c\,H_0^{-1}\int_{\bar{a}}^{1}\frac{d\bar{a}^*}{\bar{a}^{*\,2}\bar{H}(\bar{a}^*)}$, $z=\frac{1}{\bar{a}}-1$.
The luminous distance in log-scale (modulus distance) is given by
$\mu := m - M = 5\log_{10}(d_L^{\rm flat}/{\rm Mpc}) + 25$.



$$D_L(z) = cH_0^{-1}(z+1)\int_0^z \frac{dz'}{H(z')} \text{ (FOR } k=0), \; \mu = 5\log(D_L/Mpc) + 25 \; .$$



For cosmological applications it is sometimes useful to write the $f(R)$ field equations as "Einstein" field equations with a total-effective EMT: $G_{ab} = \kappa T_{ab}^{\rm tot}$

$$\kappa T_{ab}^{\rm tot} = \frac{1}{f_R}\left[f_{RR}\nabla_a\nabla_b R + f_{RRR}(\nabla_a R)(\nabla_b R) - \frac{g_{ab}}{6}\Big(Rf_R + f + 2\kappa T\Big) + \kappa T_{ab}\right] .$$

Because the EMT of matter T_{ab} itself is mixed in a non-trivial way with $f(R)$ factors, thus there is non canonical way of defining the EMT of "geometric dark energy":

$$\tilde{T}_{ab}^{X}(A,B) := AT_{ab}^{\rm tot} - BT_{ab} \quad , \tag{56}$$

Depending on the values adopted for the scalars A and B (see next slide) the defintion of the GDE EMT changes from author to author.

Alternatively, one can write $\tilde{T}^X_{ab}(A,B)$ in terms of purely geometrical quantities:

$$\tilde{T}^X_{ab}(A,B) = \kappa^{-1}\left(AG_{ab} - B\mathfrak{G}_{ab}\right) \ . \tag{57}$$

where

$$\mathfrak{G}_{ab} = f_R G_{ab} - f_{RR}\nabla_a\nabla_b R - f_{RRR}(\nabla_a R)(\nabla_b R) + g_{ab}\left[\frac{1}{2}(Rf_R - f) + f_{RR}\Box R + f_{RRR}(\nabla R)^2\right]$$

EMT of GDE	energy-density, pressure and EOS of GDE
$\tilde{T}^X_{ab}(A,B) := A T^{\rm tot}_{ab} - B T_{ab}$	$\tilde{\rho}_X = \frac{A}{\kappa f_R}\left[\frac{1}{2}(f_R R - f) - 3 f_{RR} H\dot{R} + \kappa\rho\left(1 - \frac{B f_R}{A}\right)\right]$ $\tilde{p}_X = -\frac{A}{3\kappa f_R}\left[\frac{1}{2}(f_R R + f) + 3 f_{RR} H\dot{R} - \kappa\left(\rho - 3p_{\rm rad}\frac{B f_R}{A}\right)\right]$ $\tilde{\omega}_X = \frac{\tilde{p}_X}{\tilde{\rho}_X}$

Definition	A	B	EMT
I $(T^X_{ab}, \rho_X, p_X, \omega_X)$	1	1	T^X_{ab} conserved
II $(T^{II,X}_{ab}, \rho^{II}_X, p^{II}_X, \omega^{II}_X)$	f^0_R	1	$T^{II,X}_{ab}$ conserved
III $(T^{III,X}_{ab}, \rho^{III}_X, p^{III}_X, \omega^{III}_X)$	f_R	1	$T^{III,X}_{ab}$ not conserved
IV $(T^{IV,X}_{ab}, \rho^{IV}_X, p^{IV}_X, \omega^{IV}_X)$	1	f_R^{-1}	$T^{IV,X}_{ab}$ not conserved (conserved only in vacuum)

Now, the X–EMT of the Definitions I and II are conserved ($\nabla^a T^X_{ab} = 0$) because $T^{\rm tot}_{ab}$ is conserved (due to the Bianchi identities→ $G_{ab} = \kappa T^{\rm tot}_{ab}$, $\tilde{T}^X_{ab}(A, B) := A T^{\rm tot}_{ab} - B T_{ab}$) and the EMT T_{ab} of matter alone is also conserved (Exercise). In particular, for cosmology, Definitions I and II yield

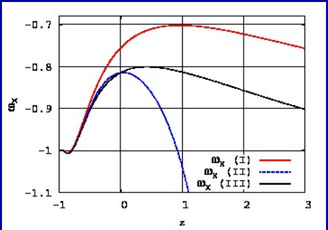
$$\dot{\rho}^i_X + 3H\left(\rho^i_X + p^i_X\right) = 0 \ . \tag{58}$$

(for $i = I, II$).
A corollary is that the EMT of Definitions III and IV are not conserved: therefore

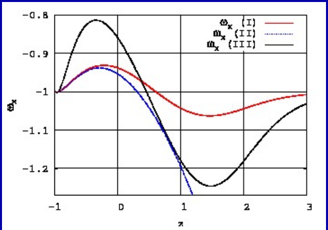
$$\dot{\rho}^i_X + 3H\left(\rho^i_X + p^i_X\right) \neq 0 \ . \tag{59}$$

(for $i = III, IV$).
This is rather unpleasant (in my opinion). Yet several authors have considered them. Furthermore, despite that the EMT of Definition II is conserved, the associated EOS in cosmology turns to be ill defined because it diverges as I will show in a moment.

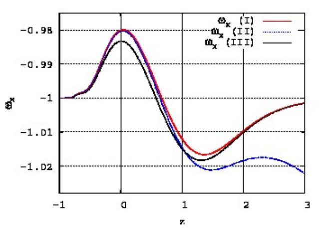
$f(R)_{MJWQ}$



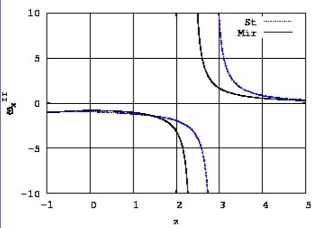
$f(R)_{St}$



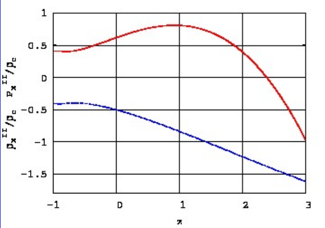
$f(R)_{HS}$



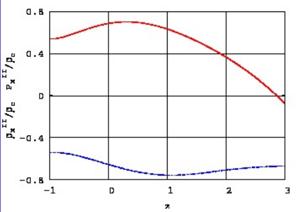
EOS II $\omega_X^{II} = \frac{p_X^{II}}{\rho_X^{II}}$: MJWQ and St



ρ_X^{II} (red) and p_X^{II} (blue) II: MJWQ



ρ_X^{II} (red) and p_X^{II} (blue) II: St

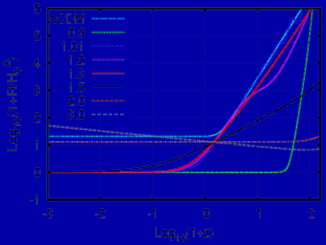


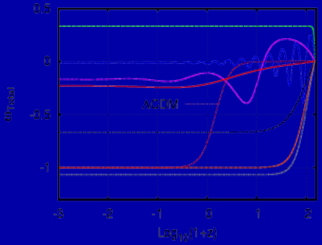
The prototype model $f(R) = \lambda R_n (R/R_n)^n$ (where $\lambda R_n = const. = \alpha_n H_0^2$, the dimensionless constant α_n is some kind of "normalization factor" which is fixed so as that for all the models, we have that $H = H_0$ today, when integrating from the matter domination epoch to the future) was one of the first to be analyzed so that it produced a late accelerated expansion. Recently it was the object of debate between several authors (S. Capozziello et al., PLB **639**, 135, 2006; PLB **664**, 135, 2008; GRG **40**, 357, 2008; Carloni et al., CQG **22**, 4839, 2005; GRG **41**, 1757, 2009) and the results of L. Amendola et al. (PRL **98**, 131302, 2007; PRD **75**, 083504, 2007; IJMPD **16**, 1555, 2007). The orange group claimed that this kind of models were ruled out because wheter the produced a late time acceleration but an inadequate matter domination epoch or the opposite. The green group criticized their analysis on two grounds: 1) They resorted to the scalar-tensor approach, which the Capozziello et al. group raised "doubts"; 2) The phase-space (dynamical system) analysis was "incomplete" (Carloni et al. group).

As concerns the first criticisms Amendola et al. repeated the analysis in the original frame and recovered the same conclusions. They have not address the second criticism.

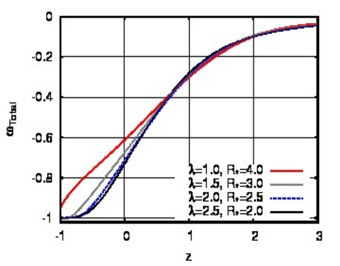
We have performed a full numerical analysis based upond the equations presented before, and we confirmed the same findings of Amendola et al., namely, these models appeared to be ruled out (L. Jaime et al., PRD **87**, 024029, 2013).

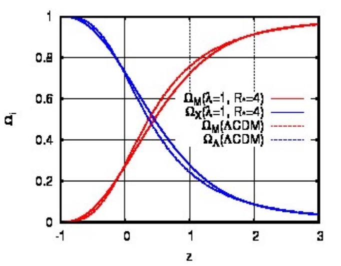
$q := -\frac{\ddot{a}}{aH^2} = -\frac{H^2+\dot{H}}{H^2} = 1 - \frac{R}{6H^2} = \frac{1}{2}(1+3\omega_{\rm tot})$. So if $\omega_{\rm tot} < -1/3$ the Universe start accelerating. The figure on the right summarized our findings:





We have also analyzed the so called *exponential gravity* model $f(R) = R_*[\tilde{R} - \lambda(1 - e^{-\tilde{R}})]$, where $\tilde{R} := R/R_*$ and $R_* \sim H_0^2$ (see arXiv:1211.0015: Proc. 100 years after Einstein in Prague). This model seems to be also cosmologically viable:





This model have been studied in more detail (perturbations) by Linder (PRD **80**, 123528, 2009) who showed that is a potentially viable model.

The following models have been ruled out in one way or another (cosmology, solar system, etc.):

$$f(R) = R - \frac{\mu^4}{R} \;, \tag{60}$$

$$f(R) = R - \frac{\mu_1^4}{R} + \mu_2^4 R \;, \tag{61}$$

$$f(R) = \alpha R^{-n} \;, \tag{62}$$

$$f(R) = R + \alpha R^{-n} \;, \; \text{(possibly viable for } \alpha < 0, n \approx 1) \tag{63}$$

$$f(R) = R^p e^{qR} \;, \tag{64}$$

$$f(R) = R^p (\log \alpha R)^2 \;, \; \text{(might succed for } p = 1, q > 0, q \neq 1) \tag{65}$$

$$f(R) = R^p e^{q/R} \;, \tag{66}$$

$$f(R) = R + \alpha R^2 \;, \tag{67}$$

In order to analyze the angular anisotropies in the CMB, and all its accompanied features, within the framework of $f(R)$ gravity, a linear perturbation analysis similar to the one of GR has to be performed. In practice, everything is more-less the same, except that instead of having an EMT of matter in the r.h.s. of the Einstein equations, one has an effective EMT that includes the geometrical parts due to the modifications of gravity. So, the perturbation procedure proceeds as follows:

$$g_{ab} = g^0_{ab} + \delta g_{ab} \quad , \quad \delta g_{ab} \ll g^0_{ab} \tag{68}$$

$$\phi = \phi^0 + \delta\phi \quad , \quad \delta\phi \ll \phi^0 \tag{69}$$

$$T_{ab} = T^0_{ab} + \delta T_{ab} \quad , \quad \delta T_{ab} \ll T^0_{ab} \quad , \tag{70}$$

where g^0_{ab} stands for the unperturbed FRW metric, and δg_{ab} is the metric perturbation which will describe the inhomogeneities and anisotropies associated with the perturbed spacetime. Here ϕ is any scalar associated with $f(R)$ gravity, like R, f_R, f_{RR}, and f_{RRR}; finally the last equation describe the pertubed EMT of matter (baryons, photons and DM, as in GR). This analysis is not new and dates back since the Starobinsky (1981) analysis of inflation and the Mukhanov *et al.* formalism (Phys. Rep. **215**, 1992). One obtains then (modulo gauges) a set of field equations for the perturbation δg_{ab} and the scalar field δR or δf_R.

Many articles treat $f(R)$ gravity like an example of a scalar-tensor theory. In that instance, the formalism of perturbations have been developed in the past. More recently, this formalism has been revisited in many articles (e.g. Hu & Sawicky, PRD, **76**, 104030, 2007; Pogosian & Silvestri, PRD; **76**, 023503, 2008) specifically for $f(R)$ gravity. Indeed, a so called *Parametrized post-Friedmannian framework* for modified gravity, was devised in the previous papers, allowing to parametrize the deviations with respect to GR-cosmology independently of the metric theory at hand. This framework is reminiscent of the *Parametrized Post-Newtonian formalism* intended to parametrize the deviations of GR with respect to other metric theories of gravity, but within the context of the solar system experiments and binary pulsar.
So, for instance, when considering only scalar metric pertubations in the Newtonian gauge around a FRW metric with Euclidean (flat) 3-slices one has

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1-2\Phi)\delta_{ij}dx^i dx^j \quad . \tag{71}$$

In the early Universe in pure GR, one has

$$\Phi = \Psi \tag{72}$$

since $\delta T^i{}_j \approx 0$ (for $i \neq j$). However, in $f(R)$ gravity, the corresponding components diagonal components of δT^{eff}_{ab} (which includes the modifications of gravity) are not zero, then $\Phi \neq \Psi$.

So one of the PPF quantities is

$$\gamma = \frac{\Phi}{\Psi} \ . \tag{73}$$

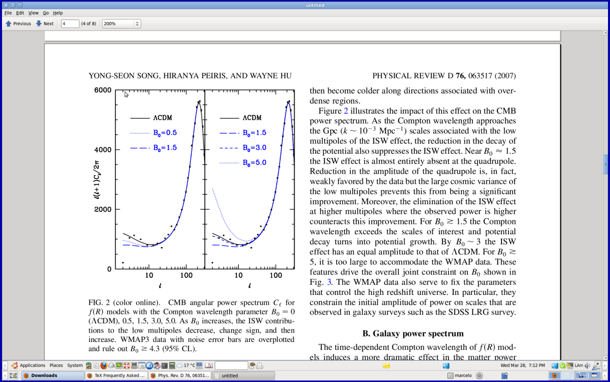
which, like in the PPN formalism, parametrize the deviations with respect to GR. The fact that in modified gravity $\gamma \neq 1$, affects the primordial (plateau) Sachs-Wolfe effect (small ℓ: large angular scales) , which is related to the CMB temperature anisotropies produced by the gravitational shifts of light when the latter traverses well potentials producen by the inhomogeneities of matter.

$$\frac{\delta T}{T}\Big|_{t_e}^{t_d} = \Phi(\vec{x_e}, t_e) - \Phi(\vec{x_d}, t_d) + \int_{t_e}^{t_d} \frac{\partial[\Phi(\vec{x}(t), t) + \Psi(\vec{x}(t), t)]}{\partial t} dt$$

where t_e = time at recombination (last scattering surface) and t_d = today
The term $\Phi(\vec{x_d}, t_d)$ gives an isotropic contribution around the observer (i.e. the probe), while the temperature anisotropies at different points of the last scattering surface $\frac{\delta T}{T}|_{t_e}$ combined with the correponding gravitational potential $\Phi(\vec{x_e}, t_e)$ gives the known term of $\Phi(\vec{x_e}, t_e)/3$. The last term corresponds to the ISW (see Merlin & Salgado, GRG **43**, 2701, 2011 for a simple and geometrical derivation)

$f(R)$ models vs CMB

From "Cosmological constraints on $f(R)$ accelerating models", Y.S. Song, H. Peiris, and W. Hu, PRD vol.**76**, 063517 (2007)



YONG-SEON SONG, HIRANYA PEIRIS, AND WAYNE HU — PHYSICAL REVIEW D **76,** 063517 (2007)

FIG. 2 (color online). CMB angular power spectrum C_ℓ for $f(R)$ models with the Compton wavelength parameter $B_0 = 0$ (ΛCDM), 0.5, 1.5, 3.0, 5.0. As B_0 increases, the ISW contributions to the low multipoles decrease, change sign, and then increase. WMAP3 data with noise error bars are overplotted and rule out $B_0 \gtrsim 4.3$ (95% CL).

then become colder along directions associated with overdense regions.

Figure 2 illustrates the impact of this effect on the CMB power spectrum. As the Compton wavelength approaches the Gpc ($k \sim 10^{-3}$ Mpc^{-1}) scales associated with the low multipoles of the ISW effect, the reduction in the decay of the potential also suppresses the ISW effect. Near $B_0 \approx 1.5$ the ISW effect is almost entirely absent at the quadrupole. Reduction in the amplitude of the quadrupole is, in fact, weakly favored by the data but the large cosmic variance of the low multipoles prevents this from being a significant improvement. Moreover, the elimination of the ISW effect at higher multipoles where the observed power is higher counteracts this improvement. For $B_0 \gtrsim 1.5$ the Compton wavelength exceeds the scales of interest and potential decay turns into potential growth. By $B_0 \sim 3$ the ISW effect has an equal amplitude to that of ΛCDM. For $B_0 \gtrsim 5$, it is too large to accommodate the WMAP data. These features drive the overall joint constraint on B_0 shown in Fig. 3. The WMAP data also serve to fix the parameters that control the high redshift universe. In particular, they constrain the initial amplitude of power on scales that are observed in galaxy surveys such as the SDSS LRG survey.

B. Galaxy power spectrum

The time-dependent Compton wavelength of $f(R)$ models induces a more dramatic effect in the matter power

Solar system tests: weak field limit. Consider static and spherically symmetric perturbations ($|\phi|, |\psi| \ll 1$) around a De Sitter background:

$$ds^2 = -(1 - \phi - \Lambda_{eff} r^2)dt^2 + (1 + \psi - \Lambda_{eff} r^2)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (74)$$

In GR+Λ

$$\phi = 2M/r \quad (75)$$
$$\psi = 2M/r \quad (76)$$
$$\Lambda_{eff} = \Lambda \quad (77)$$
$$\gamma = \frac{\psi}{\phi} = 1 \quad (78)$$

where γ is one of the Post-Newtonian parameters. At solar system scales we can in fact neglect the term $\Lambda_{eff} r^2$. Now, in $f(R)$ gravity

$$\phi = 2M/r \quad (79)$$
$$\psi = 2\gamma M/r \quad (80)$$
$$\Lambda_{eff} = R_1/4 \quad (81)$$
$$\gamma \neq 1 \quad (82)$$

In fact γ depends on the the parameters of the theory $f(R)$ and on the global properties of the Sun, like $R_\odot$ and $M_\odot$. According to the observations (Cassini probe: Bertotti *et al.* Nature **425**, 2003, 474)

$$|\gamma - 1| \sim 10^{-5} \tag{83}$$

It turns out that (Faulkner et al., PRD **76**, 063505, 2007)

$$|\gamma - 1| = \frac{2\Delta}{3+\Delta} \tag{84}$$

where Δ is the so-called *thin shell parameter* which is related to the *chameleon*: (Khoury & Weltman, PRL **93**, 171104, 2004); PRD **69**, 044026, 2004) the scalar field degree of freedom f_R is supressed in regions of “high” density (the Sun) and at low density (cosmological scales) has noticeable effects, like the cosmic acceleration. This phenomenon is highly dependent on the contrast density between the central object and the surrounding environment and also on the details of the specific $f(R)$ theory. When the chamelon effect takes place, the scalar field f_R behaves like the electric potential within a conductor: inside the object $f_R \approx const.$ except within a thin shell $\delta R_\odot$ with $\Delta = \delta R_\odot / R_\odot \ll 1$, where the gradient of f_R is large (*screening effect* like within a conductor).

Outside the object $f_R \propto M_\odot/r$. In this instance it is possible to satisfy the bound (Faulkner et al., PRD **76**, 063505, 2007)

$$|\gamma - 1| = \frac{2\Delta}{3+\Delta} < 10^{-5} \tag{85}$$

if $\Delta = \delta R_\odot / R_\odot \ll 1$. The thin shell parameter depends on the two minima of the **effective potential** $V_{\rm eff}(f_R, \rho_{\rm in,out})$ whose respective values inside the extended object (e.g. the Sun) where and outside depend on $\rho_{\rm in}$ and $\rho_{\rm out}$ and the bulk properties of the object (e.g. $M_\odot, R_\odot$).
However, when the chameleon does not ensue, f_R behaves like the electric potential within a dielectric: it has important variations within the object and the “thin” shell disapears: $\Delta = \delta R_\odot / R_\odot \sim 1$ and therefore

$$|\gamma - 1| = \frac{2\Delta}{3+\Delta} \sim \frac{1}{2} >> 10^{-5} \tag{86}$$

$f(R)$ Models

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mosphere in the high-curvature regime.

The numerical solutions shown in Fig. 6 verify the qualitative behavior described in the previous section. For $|f_{R0}| \lesssim 10^{-2}$, the deviations from the high-curvature $R = \kappa^2\rho$ limit are fractionally small since the Compton condition is everywhere satisfied. For $|f_{R0}| \lesssim 10^{-1}$, the break to low curvature occurs in the corona. This break occurs gradually in the field profile $f_R(r)$ but rapidly in the curvature (see Fig. 7). At small-field values, a small change in f_R represents a large change in the curvature. $M_{\rm eff}$ is approximately just the mass between this transition and the point at which the galactic density exceeds the corona. Outside of this transition, the exterior field relaxes to the galactic value as $|\Delta f_R| \propto e^{-m_{f_R} r}/r$. In these examples, the Compton wavelength in the galaxy is of order 10^3–$10^4 r_\odot$ and the mass term further suppresses the deviations from GR.

In Fig. 8 we show $|\gamma - 1|$ for the same $n = 4$ models. The deviations peak at $\sim 10^{-15}$. Such deviations easily pass the stringent solar-system tests of gravity from the Cassini mission [98]

$$|\gamma - 1| < 2.3 \times 10^{-5}, \qquad (63)$$

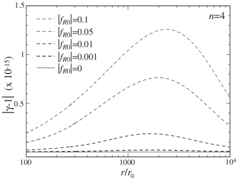

FIG. 8. Metric deviation parameter $|\gamma - 1|$ for $n = 4$ models and a series of cosmological field amplitudes f_{R0} with a galactic field that minimizes the potential. These deviations are unobservably small for the whole range of amplitudes.

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- In fact it was only recently that several authors have tried to construct relativistic extended objects in the framework of $f(R)$ gravity.
- In particular **Kobayashi and Maeda** (PRD **78**, 064019, 2008; PRD **79**, 024009, 2009) using the Starobinsky model $f(R) = R - \lambda R_* \left\{1 - \left[1 + (R/R_*)^2\right]^{-\beta}\right\}$ shown that such objects cannot be constructed because a curvature singularity developed within the object.
- Later **Babichev and Langlois** (PRD **80**, 121501(R) 2009; gr-qc/0911.1297, 2010) reanalyzed the issue and concluded that KM results was a consequence of the use of an incompressible fluid, and that using a more realistic EOS (polytropes) such singularities were not found.
- However, **Upadhye and Hu** (PRD **80**, 064002,2009) found that relativistic extended objects can indeed be constructed, but that the absence of singularities got nothing to do with the EOS, but rather with a "chameleon mechanism".

- The common feature of the aforementioned works is that authors used the above mapping to constuct such objects.
- Under such mapping the Ricci scalar has a behavior of the sort $R \sim 1/(\chi - \chi_0)$ where $\chi := \partial_R f$ and $\chi_0 = \text{const.}$
- The key point is to determine if the dynamics of χ leads it or not to the value $\chi = \chi_0$ within the spacetime generated by the relativistic object.
- Irrespective of the different results and confusing explanations obtained by those works we argue that their conclusions are rather questionable due to the fact that the above scalar-field variables are ill defined. To be more specific, the scalar-field potential used to study the dynamics of χ is not single valued and possesses pathological features. Since similar kind of singularities were also found in the cosmological setting (Frolov, PRL **101**, 061103, 2008), it is then worrisome that the ill-defined potential play such a crucial role in those analyses (several authors have already criticized the use of such potentials).

Some exact solutions (a small detour)

We consider the following metric that allows us to describe SSS

$$ds^2 = -n(r)dt^2 + m(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{87}$$

The field Eqs. then read

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The field Eqs. then read

$$R'' = \frac{1}{3f_{RR}}\left[m(\kappa T + 2f - Rf_R) - 3f_{RRR}R'^2\right] + \left(\frac{m'}{2m} - \frac{n'}{2n} - \frac{2}{r}\right)R' . \quad (88)$$

(where $' := d/dr$). From the $t - t$, $r - r$, and $\theta - \theta$ of field Eqs. and after several non-trivial manipulations we found

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(where $'$:= d/dr). From the $t - t$, $r - r$, and $\theta - \theta$ of field Eqs. and after several non-trivial manipulations we found

$$\begin{aligned} m' &= \frac{m}{r(2f_R + rR'f_{RR})}\Big\{2f_R(1-m) - 2mr^2\kappa T^t{}_t + \frac{mr^2}{3}(Rf_R + f + 2\kappa T) + \frac{rR'f_{RR}}{f_R}\Big[\frac{mr^2}{3}(2Rf_R - f + \kappa T) \\ &\quad -\kappa mr^2(T^t{}_t + T^r{}_r) + 2(1-m)f_R + 2rR'f_{RR}\Big]\Big\} , \end{aligned} \tag{89}$$

$$n' = \frac{n}{r(2f_R + rR'f_{RR})}\left[mr^2(f - Rf_R + 2\kappa T^r{}_r) + 2f_R(m-1) - 4rR'f_{RR}\right] , \tag{90}$$

$$n'' = \frac{2nm}{f_R}\left[\kappa T^\theta{}_\theta - \frac{1}{6}(Rf_R + f + 2\kappa T) + \frac{R'}{rm}f_{RR}\right] + \frac{n}{2r}\left[2\left(\frac{m'}{m} - \frac{n'}{n}\right) + \frac{rn'}{n}\left(\frac{m'}{m} + \frac{n'}{n}\right)\right] . \tag{91}$$

Remarks of the above system of ODE's:

- Notice that Eqs. for n' and n'' are not independent. In fact, one has the freedom of using one or the other. Nevertheless, we have used both to check the consistency of our equations and the numerical code (solutions).
- Now, from the usual expression of R in terms of the Christoffel symbols one obtains,

$$\begin{aligned} R &= \frac{1}{2r^2n^2m^2}\Big[4n^2m(m-1) + rnm'(4n+rn') \\ &\quad -2rnm(2n'+rn'') + r^2mn'^2\Big] . \end{aligned} \tag{92}$$

  As one can check by a direct calculation, that using the Eqs. for m', n', n'' in the above Eq., one finds an identity $R \equiv R$. This result confirms two things: 1) Our Eqs. are consistent and no elementary mistake was made in their derivation; 2) The previous expression for R does not provide any further information.
- When defining the first order variables $Q_n = n'$ and $Q_R := R'$, the above system of ODE's have the form $dy^i/dr = \mathcal{F}^i(r, y^i)$ where $y^i = (m, n, Q_n, R, Q_R)$ and therefore can be solved numerically. As far as we are aware, such a system has not been considered previously. These equations can be used to tackle several aspects of SSS spacetimes in $f(R)$ gravity.

- We observe that for $f(R) = R$ our system of ODE's reduce to the well known equations of GR for SSS spacetimes.

- Like in the general case, our system of ODE's in vacuum has the exact de Sitter solution $n(r) = m(r)^{-1} = 1 - \Lambda_{\rm eff} r^2/3$, $R = R_1 = {\rm const.}$ with $\Lambda_{\rm eff} = R_1/4$ and $R_1 = 2f(R_1)/f_R(R_1)$.

- We also need the matter equations $\nabla^a T_{ab} = 0$. So for $T_{ab} = (\rho + p)u_a u_b + g_{ab}p$, we get

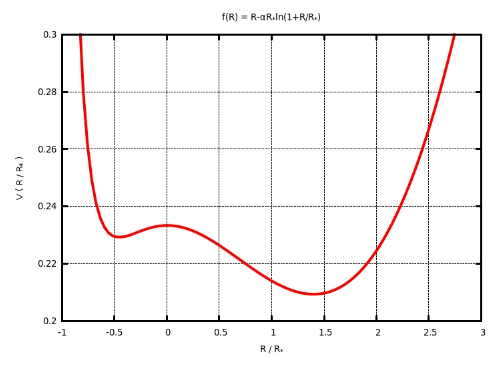
$$p' = -(\rho + p)n'/2n \tag{93}$$

This is the modified Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium which is to be complemented by an EOS.

- For simplicity we shall assume an incompressible fluid ($\rho = const.$)
- We integrate the equations numerically outwards from the origin $r = 0$ and impose regularity conditions at $r = 0$. We fix $p(0)$ (the central pressure) to a given value (this fixes one "star" configuration).
- We obtain $R(0)$ by a shooting method so that asymptotically the solution matches the de Sitter solution $R = R_1 = const.$ where R_1 is a critical point of the "potential" $V(R) = -Rf(R)/3 + \int^R f(x)dx$. That is, R_1 is a point where $dV(R)/dR = (2f - f_R R)/3$ vanishes.
- This potential is radically different from the scalar-field potential that arises under the STT map. Furthermore, $V(R)$ is as well defined as the function $f(R)$ itself.

- We used first the model $f(R) = R - \alpha R_* \ln(1 + R/R_*)$ (α, R_* are positive constants; R_* sets the scale) proposed by **Miranda et. al.** (PRL **102**, 221101, 2009). Warning: for this model f_R is not positive definite in general (only if $\alpha < 1 + R/R_*$) but $f_{RR} > 0$. Note also that $f(R)$ is only well defined for $0 < 1 + R/R_*$. A priori there is no guarantee that solutions for R exist while satisfying such conditions.
- Those authors mapped the theory to the STT counterpart. However, unlike the Starobinsky model (see below), in this case the resulting scalar-field potential turns to be single-valued. This is why we take it in order to compare (calibrate) directly with our method.
- They did not find any singularity within the object.
- Under our approach we associate to this $f(R)$ the potential $V(R) = \frac{R_*^2}{6}\left\{(1+\tilde{R})\left[\tilde{R} + (6\alpha - 1)\right] - 2\alpha(3 + 2\tilde{R})\ln\left[1 + \tilde{R}\right]\right\}$, where $\tilde{R} = R/R_*$. For $\alpha = 1.2$ (the value that Miranda et al. assumed) this potential has several critical points

Some exact solutions (a small detour)



PRL **102**, 221101 (2009)

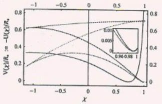
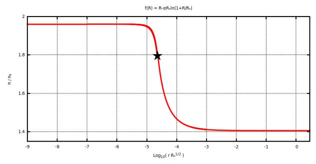
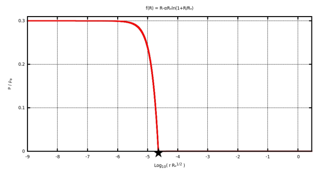
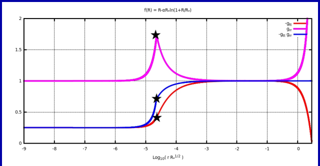


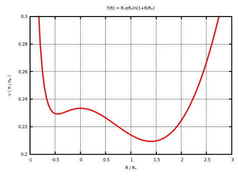
FIG. 1 (color online). $V(\chi)/R_* := -U(\chi)/R_*$ for different models: ours, with $\alpha = 2$ (blue solid line), Starobinski's for $\{n = 2, \lambda = 2\}$ [see [9]] (red dashed line), and Hu and Sawicki's for $\{n = 2, m^2 = 1, c_1/c_2 = 2\}$ [see [8]] (green dot-dashed line). The physically interesting region is $0 < \chi < 1$. For the multi-valued potentials only the lower lines are physical.

We have used our approach and found no singularities whatsoever in compact objects (like in Miranda et al.)

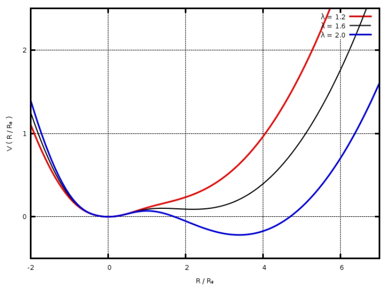




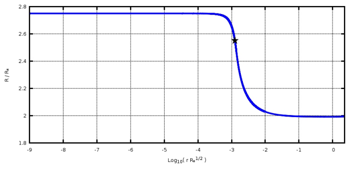


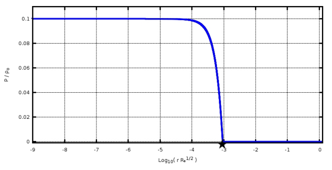


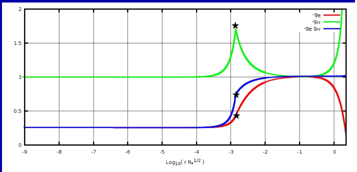
- We then used the Starobinsky model $f(R) = R - \lambda R_* \left\{1 - \left[1 + (R/R_*)^2\right]^{-\beta}\right\}$ with $\beta = 1$. The controversy about the existence of extended relativistic objects (or absence thereof) was originated using this model. As we saw, the STT approach gives rise to multivalued potentials.
- With our method, the potential is given by $(\tilde{R} = R/R_*)$
$$V(R) = \frac{R_*^2}{3}\left\{\frac{\tilde{R}}{2}\left[\tilde{R} - 4\lambda - 2\lambda\left(1+\tilde{R}^2\right)^{-1}\right] - 3\lambda \arctan\left(\tilde{R}\right)\right\}.$$
- This potential has a rich structure depending on the value of λ.

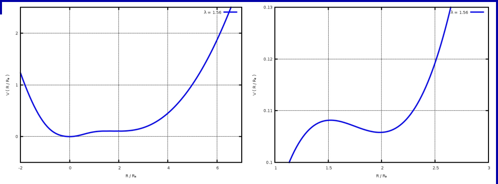


For the value $\lambda = 1.56$ we found the following solutions using a shooting method aiming to the local minimum. No singularities whatsoever where found.









Discussion:

Although we have not find singularities in the spacetime generated by this compact objects, there is a caveat in the above construction:
In almost all $f(R)$ models the dimension parameters like R_*, which settles the scale $\sim D^{-2}$, are chosen such that $R_*/G_0 \sim \tilde{\Lambda} \sim 10^{-29}\mathrm{g\,cm}^{-3}$ $(\tilde{\Lambda} = \Lambda/G_0)$. On the other hand, if one wants to build a realistic neutron star, the typical densities at the center are $\rho(0) \sim \rho_{\rm nuc} \sim 10^{14}\mathrm{g\,cm}^{-3}$. That is, there are around 43 orders of magnitude between the typical density within a neutron star and the average density of the Universe! This ratio between densities naturally appears in the equations since the parameters which define the specific $f(R)$ theory are of the order of $\tilde{\Lambda}$, while the appropriate dimension within neutron stars is $\rho_{\rm nuc}$. So in units of $\rho_{\rm nuc}$, the cosmological constant turns out to be ridiculously small, while in units of $\tilde{\Lambda}$, $\rho(r)$ and $p(r)$ turn out to be ridiculously large within the neutron star.

In other words, the scale of a neutron star is $\sim \mathrm{km}$ while the cosmological scales are $\sim 100\mathrm{Mpc}$. We, like the other authors, have not solved this technical problem...YET, and therefore have constructed "compact" objects which whether are realisitic but then $\tilde{\Lambda}$ is not. Or the opposite. In both cases the objects are compact and relativistic in the sense that $p \sim \rho$ and $G_0\mathcal{M}/\mathcal{R}$ is not far from $4/9$, where $\mathcal{M}$ is the "ADM" mass defined in asymptotically de Sitter spacetimes.

Conclusions

- $f(R)$ theories are alternative theories of gravity that can produce an accelerated expansion of the universe "without" the introduction of Λ. Some specific $f(R)$ models can pass several gravitational tests (e.g. the Solar System tests, cosmological). They have some predictions different from GR+ Λ (e.g. variable EOS of dark energy, new gravitational-wave modes – breathing mode –, different Sachs-Wolfe effect, ...)
- However, in my opinion they introduce more troubles than solutions. There is no fundamental principle that allows to single out one function $f(R)$. Simplicity favors: $f(R) = R - 2\Lambda$ (i.e. GR+ Λ). Time will tell if models different from GR will be taken seriously in the future.
- As concerns the EOS of dark-energy within $f(R)$ gravity ambiguities may arise. Be aware of them ! Putting aside this issue, further experiments will determine if such EOS varies in cosmic time or not (e.g BigBOSS–DESI–, EUCLID, PanSTARR, WFIRST, etc.).

Summary of standard inflation

In one of the simples scenarios, (primordial) inflation (i.e. the primordial accelerated expansion of the Universe, as opposed to the late acceleration expansion $\rightarrow$ SNIa) is produced by a single scalar field (as opposed to multiple scalar fields, or a modification of gravity $\rightarrow$ Starobinsky inflation $f(R) = R + \alpha R^2$; I'll come back to this point later) in the so called roll-over approximation:

$$H^2 = \frac{\kappa}{3}\left(\frac{1}{2}\dot{\phi}_0^2 + V(\phi_0)\right) = \approx \frac{\kappa}{3} V(\phi_0) \; , \tag{94}$$

where $\kappa := 8\pi G$. That is we assume

$$\dot{\phi}_0^2 \ll V(\phi_0) \; , \tag{95}$$

the potential dominates over the kinetic term.

As concerns the KG equation, one has

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V'(\phi_0) = 0 \tag{96}$$

where we assume

$$\ddot{\phi}_0 \ll 3H\dot{\phi}_0 \; , \tag{97}$$

the friction dominates over the acceleration term.

The roll-over parameters are given by

$$\epsilon := \frac{\kappa}{2}\left(\frac{\dot{\phi}_0}{H}\right)^2 , \quad (98)$$

$$\approx \frac{1}{2\kappa}\left(\frac{V'(\phi_0)}{V(\phi_0)}\right)^2 , \quad (99)$$

$$\eta := -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} , \quad (100)$$

such that

$$\epsilon \ll 1 \ , \ \eta \ll 1 \quad (101)$$

When analysing the origin of perturbations due the quantum fields, it is better to work with the conformal time τ:

$$ds^2 = \bar{a}^2(\tau)\left[-d\tau^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \quad (102)$$

$$= \bar{a}^2(\tau)\left[-d\tau^2 + \delta_{ij}dx^i dx^j\right] \quad (103)$$

where $\tau = \int dt/a(t)$, $\bar{a}(\tau) = a(t(\tau))$ (in the following we drop the overline in all the variables that depend on conformal time).

When considering perturbations we assume that the perturbed scalar-field is given by

$$\Phi(\tau,\vec{x}) = \phi_0(\tau)+\phi(\tau,\vec{x}) \ , \quad (104)$$

$$\phi(\tau,\vec{x}) \ll \phi_0(\tau) \quad (105)$$

where $\phi_0(\tau)$ is the background (unperturbed) scalar field and $\phi(\tau,\vec{x})$ is a perturbation.

In order to consider quantum effects we promote $\phi(\tau,\vec{x}) \to \hat{\phi}(\tau,\vec{x})$, where $\hat{\phi}(\tau,\vec{x})$ is the scalar-field operator acting upon quantum states defined on a Hilbert space (the generalization of fock space in a curved spacetime). So the field is decomposed in Fourier modes:

$$\hat{\phi}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\phi_{\vec{k}}(\tau)\hat{b}_{\vec{k}} e^{\imath\vec{k}\cdot\vec{x}} + \phi^*_{\vec{k}}(\tau)\hat{b}^\dagger_{\vec{k}} e^{-\imath\vec{k}\cdot\vec{x}}\right] \ . \quad (106)$$

where the creation and anhilation operators obey the commutation rules

$$[\hat{b}_{\vec{k}},\hat{b}^\dagger_{\vec{k}'}] = \delta^3(\vec{k}-\vec{k}') \quad (107)$$

Bunch-Davis vacuum $\hat{b}_{\vec{k}}|0_b>=0$.

The equation of motion that satisfies the coefficients $\phi_{\vec{k}}(\tau)$ is

$$\phi''_{\vec{k}}(\tau) + 2\frac{a'}{a}\phi'_{\vec{k}}(\tau) + k^2\phi_{\vec{k}}(\tau) = 0 \tag{108}$$

Or in terms of the new variable

$$u_{\vec{k}} = a(\tau)\phi_{\vec{k}} \quad , \tag{109}$$

$$u''_{\vec{k}}(\tau) + \left(k^2 - \frac{a''}{a}\right) u_{\vec{k}}(\tau) = 0 \tag{110}$$

There are two limits at which this equation can be solved exactly:
Short-wave limit (ultraviolet limit) (SWL) (Minkowski limit): $a''/a \ll k$

$$u''_{\vec{k}}(\tau) + k^2 u_{\vec{k}}(\tau) = 0 \tag{111}$$

Then, the normalized solution is in general

$$u_{\vec{k}}(\tau) = \frac{1}{\sqrt{k}}\left[A_{\vec{k}}e^{-\imath\tau} + B_{\vec{k}}e^{\imath\tau}\right] \tag{112}$$

Bunch-Davis vacuum $A_{\vec{k}} = 1$, $B_{\vec{k}} = 0$.

Long-wave limit (mode freezing) (LWL): $k \ll a''/a$

$$u''_{\vec{k}}(\tau) - \frac{a''}{a} u_{\vec{k}}(\tau) = 0 \tag{113}$$

The normalized solution is

$$u_{\vec{k}}(\tau) = \frac{1}{\sqrt{2k}} \left(\frac{-\imath}{k\tau} \right) = \frac{1}{\sqrt{2k}} \left(\frac{aH}{k} \right) , \tag{114}$$

$$|\phi_{\vec{k}}(\tau)| = \frac{|u_{\vec{k}}(\tau)|}{a} = \frac{H}{\sqrt{2}k^{3/2}} \tag{115}$$

where $\tau = -1/[aH(1-\epsilon)]$. Remember $\epsilon \ll 1$. The simplest case $\epsilon = 0$ gives $\tau = -1/[aH] = -d_H$, where d_H, is the comoving distance at the horizon. The power spectrum is defined in terms of the 2-point correlation function

$$< 0|\hat{\phi}(\tau,\vec{x})\hat{\phi}(\tau,\vec{x}')|0 > = \int \frac{d^3k}{(2\pi)^{3/2}} |\frac{u_{\vec{k}}(\tau)}{a}|^2 e^{\imath\vec{k}\cdot(\vec{x}-\vec{x}')} = \int \frac{dk}{k} P(k) e^{\imath\vec{k}\cdot(\vec{x}-\vec{x}')} , \tag{116}$$

$$\Longrightarrow P(k) = \frac{k^3}{2\pi^2} |\frac{u_{\vec{k}}(\tau)}{a}|^2 \tag{117}$$

In the LWL one has

$$P(k) \approx \left(\frac{H}{2\pi}\right) = \sqrt{\frac{\kappa V}{12\pi^2}} \approx const. \tag{118}$$

That is, the primordial power spectrum $P(k)$ is approximatively scale invariant. The number of e-folds $N = -\ln a$, with $a = e^{-N}$ and $H = -dN/dt$, lead to

$$N = -\int H dt = -\sqrt{\kappa}\int \frac{1}{\sqrt{\epsilon(\phi)}} = -\kappa \int^{\phi} \frac{V(\phi)}{V'} d\phi \ , \tag{119}$$

so $N = N(\phi)$, and then $\phi(N)$, and therefore $N_{\vec{k}} = N(\phi_{\vec{k}})$. So

$$P^{1/2}(k) \approx \sqrt{\frac{\kappa V(N_{\vec{k}})}{12\pi^2}} \tag{120}$$

$N_{\vec{k}}$ is the number of e-folds of mode k at the horizon $k = aH$. A better approximation of all this consists in considering $\epsilon \ll 1$ but $\epsilon = const. \neq 0$. Then $u_{\vec{k}}(\tau)$ is given in terms of Bessel functions, and the spectrum depends slightly on k: $P(k) \approx k^n$, $n \approx -2\epsilon$. And even a more accurate description is when $\epsilon(\phi)$. But for our purpose it's sufficient to consider the above approximations.

Now, at the linear limit we have mainly two kind of perturbations for the metric and the matter fields: scalar and tensor.
The tensor perturbations associates with the metric is

$$\delta g_{ij} = \sqrt{4\kappa}(\phi_+ \hat{e}^+_{ij} + \phi_x \hat{e}^x_{ij}) \tag{121}$$

At the quantum level one can prove that the tensor perturbations ϕ_+ and ϕ_x can be treated as two scalars. Therefore, the power spectrum due to the tensor modes is

$$P_T \sim 2 \times const. \left(\frac{H}{2\pi}\right)^2 \tag{122}$$

$$= \frac{2\kappa H^2}{\pi^2} , \tag{123}$$

The power spectrum for scalar perturbations in the matter and the metric (the "Newtonian" and/or post-Newtonian potentials Φ and Ψ) is

$$P_S \sim const. \left(\frac{H}{2\pi}\right)^2 \tag{124}$$

$$= \frac{2\kappa H^2}{8\pi^2 \epsilon} , \tag{125}$$

Therefore the ratio bewteen the above power-spectrum is given by

$$r := \frac{P_T}{P_S} \approx 16\epsilon \ , \tag{126}$$

For instance, taking Linde's model $V(\phi) = \lambda\phi^4$ and inserting this into the slow-roll parameter ϵ

$$\epsilon = \frac{1}{2\kappa}\left(\frac{V'(\phi_0)}{V(\phi_0)}\right)^2 \tag{127}$$

and using the relationship between ϕ_0 and N (e-folds), one obtains

$$r = \frac{16}{N} \ . \tag{128}$$

For instance with $N \approx 60$, one gets $r \approx 0.25$.

Now, all this has been done in the framework of a single-field inflation and within GR. What as I mentioned before, $f(R)$ theories can produce a late-time accelerated expansion. Can one use this alternative theory to produce an early inflationary period "without" introducing explicitly a scalar-field ?. The answer is YES ! → Starobinsky model $f(R) = R + \alpha R^2$. One can re-write $f(R)$ theories into a kind of scalar-tensor theories (STT) (Jordan or Einstein frames) where one obtains a scalar-field potential $V(\phi)$ where $\phi = f_R$.
But in addition, STT and/or $f(R)$ theories produce an extra degree of freedom which produce scalar-gravitational waves ("breathing" mode). Therefore, one expects that the two modified power spectrums $P_T^{Mod}(f_{RR})$ and $P_S^{Mod}(f_{RR})$, will depend on f_{RR}. So that in pure GR $f(R) = R$ (without an extra scalar-field) the two power spectrums vanish. Then measuring the ratio $r(f_{RR})$ can constraint the form of $f(R)$. As mentioned, the Starobinsky model is just one example that can constraint the value α.

Conclusions

- $f(R)$ theories are alternative theories of gravity that can produce an accelerated expansion of the universe "without" the introduction of Λ. Some specific $f(R)$ models can pass several gravitational tests (e.g. the Solar System tests). They have some predictions different from GR+ Λ (e.g. variable EOS of dark energy, new gravitational-wave modes – breathing mode ––, different Sachs-Wolfe effect, ...)
- However, in my opinion they introduce more troubles than solutions. There is no fundamental principle that allows to single out one function $f(R)$. Simplicity favors: $f(R) = R - 2\Lambda$ (i.e. GR+ Λ). Time will tell if models different from GR will be taken seriously in the future.
- The are other issus concerning the EOS associated with $f(R)$ as geometric dark energy (however, I didn't have time to discuss them). Observationally further experiments will determine if such EOS is vaiable or not (e.g BigBOSS–DESI–, EUCLID, PanSTARR, WFIRST, etc.).
- Finally, the BICEP, PLANCK or other future experiments can constraint the inflationary model. In particular, the inflationary models arising from modifications of gravity $\rightarrow$ an extra scalar degree of freedom propagates and it affects the tensor-to-scalar-spectrum-amplitudes ratio.

- Among the alternative theories of gravity proposed to solve this "crisis" we briefly analyzed $f(R)$ gravity, which has been one of the most popular proposals in recent years and discussed some of its drawbacks and merits.

- Among the alternative theories of gravity proposed to solve this "crisis" we briefly analyzed $f(R)$ gravity, which has been one of the most popular proposals in recent years and discussed some of its drawbacks and merits.
- We also proposed a much more robust approach to treat them and proved its usefulness in the context of cosmology and compact extended bodies.

- In fact it was only recently that several authors have tried to construct relativistic extended objects in the framework of $f(R)$ gravity.
- In particular **Kobayashi and Maeda** (PRD **78**, 064019, 2008; PRD **79**, 024009, 2009) using the Starobinsky model $f(R) = R - \lambda R_* \left\{1 - \left[1 + (R/R_*)^2\right]^{-\beta}\right\}$ shown that such objects cannot be constructed because a curvature singularity developed within the object.
- Later **Babichev and Langlois** (PRD **80**, 121501(R) 2009; gr-qc/0911.1297, 2010) reanalyzed the issue and concluded that KM results was a consequence of the use of an incompressible fluid, and that using a more realistic EOS (polytropes) such singularities were not found.
- However, **Upadhye and Hu** (PRD **80**, 064002,2009) found that relativistic extended objects can indeed be constructed, but that the absence of singularities got nothing to do with the EOS, but rather with a "chameleon mechanism".

- The common feature of the aforementioned works is that authors used the above mapping to constuct such objects.
- Under such mapping the Ricci scalar has a behavior of the sort $R \sim 1/(\chi - \chi_0)$ where $\chi := \partial_R f$ and $\chi_0 = \mathrm{const.}$
- The key point is to determine if the dynamics of χ leads it or not to the value $\chi = \chi_0$ within the spacetime generated by the relativistic object.
- Irrespective of the different results and confusing explanations obtained by those works we argue that their conclusions are rather questionable due to the fact that the above scalar-field variables are ill defined. To be more specific, the scalar-field potential used to study the dynamics of χ is not single valued and possesses pathological features. Since similar kind of singularities were also found in the cosmological setting (Frolov, PRL **101**, 061103, 2008), it is then worrisome that the ill-defined potential play such a crucial role in those analyses (several authors have already criticized the use of such potentials).

We consider the following metric that allows us to describe SSS

$$ds^2 = -n(r)dt^2 + m(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \quad (129)$$

The field Eqs. then read

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The field Eqs. then read

$$R'' = \frac{1}{3f_{RR}}\left[m(\kappa T + 2f - Rf_R) - 3f_{RRR}R'^2\right] + \left(\frac{m'}{2m} - \frac{n'}{2n} - \frac{2}{r}\right)R' \; . \tag{130}$$

(where $' := d/dr$). From the $t-t$, $r-r$, and $\theta-\theta$ of field Eqs. and after several non-trivial manipulations we found

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(where $'$ $:= d/dr$). From the $t-t$, $r-r$, and $\theta-\theta$ of field Eqs. and after several non-trivial manipulations we found

$$\begin{aligned} m' &= \frac{m}{r(2f_R + rR'f_{RR})}\Big\{2f_R(1-m) - 2mr^2\kappa T^t{}_t + \frac{mr^2}{3}(Rf_R + f + 2\kappa T) + \frac{rR'f_{RR}}{f_R}\Big[\frac{mr^2}{3}(2Rf_R - f + \kappa T) \\ &\quad -\kappa mr^2(T^t{}_t + T^r{}_r) + 2(1-m)f_R + 2rR'f_{RR}\Big]\Big\} , \end{aligned} \tag{131}$$

$$n' = \frac{n}{r(2f_R + rR'f_{RR})}\left[mr^2(f - Rf_R + 2\kappa T^r{}_r) + 2f_R(m-1) - 4rR'f_{RR}\right] , \tag{132}$$

$$n'' = \frac{2nm}{f_R}\left[\kappa T^\theta{}_\theta - \frac{1}{6}(Rf_R + f + 2\kappa T) + \frac{R'}{rm}f_{RR}\right] + \frac{n}{2r}\left[2\left(\frac{m'}{m} - \frac{n'}{n}\right) + \frac{rn'}{n}\left(\frac{m'}{m} + \frac{n'}{n}\right)\right] . \tag{133}$$

Remarks of the above system of ODE's:

- Notice that Eqs. for n' and n'' are not independent. In fact, one has the freedom of using one or the other. Nevertheless, we have used both to check the consistency of our equations and the numerical code (solutions).
- Now, from the usual expression of R in terms of the Christoffel symbols one obtains,

$$R = \frac{1}{2r^2 n^2 m^2}\Big[4n^2 m(m-1) + rnm'(4n + rn') - 2rnm(2n' + rn'') + r^2 mn'^2\Big] \,. \tag{134}$$

  As one can check by a direct calculation, that using the Eqs. for m', n', n'' in the above Eq., one finds an identity $R \equiv R$. This result confirms two things: 1) Our Eqs. are consistent and no elementary mistake was made in their derivation; 2) The previous expression for R does not provide any further information.
- When defining the first order variables $Q_n = n'$ and $Q_R := R'$, the above system of ODE's have the form $dy^i/dr = \mathcal{F}^i(r, y^i)$ where $y^i = (m, n, Q_n, R, Q_R)$ and therefore can be solved numerically. As far as we are aware, such a system has not been considered previously. These equations can be used to tackle several aspects of SSS spacetimes in $f(R)$ gravity.

- We observe that for $f(R) = R$ our system of ODE's reduce to the well known equations of GR for SSS spacetimes.

- Like in the general case, our system of ODE's in vacuum has the exact de Sitter solution $n(r) = m(r)^{-1} = 1 - \Lambda_{\rm eff} r^2/3$, $R = R_1 = {\rm const.}$ with $\Lambda_{\rm eff} = R_1/4$ and $R_1 = 2f(R_1)/f_R(R_1)$.

- We also need the matter equations $\nabla^a T_{ab} = 0$. So for $T_{ab} = (\rho + p)u_a u_b + g_{ab}p$, we get

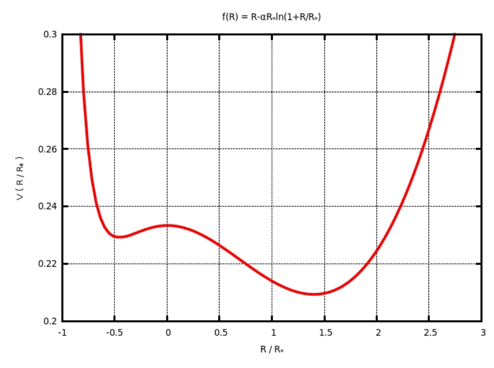
$$p' = -(\rho + p)n'/2n \qquad (135)$$

  This is the modified Tolman-Oppenheimer-Volkoff equation of hydrostatic equilibrium which is to be complemented by an EOS.

- For simplicity we shall assume an incompressible fluid ($\rho = const.$)
- We integrate the equations numerically outwards from the origin $r = 0$ and impose regularity conditions at $r = 0$. We fix $p(0)$ (the central pressure) to a given value (this fixes one "star" configuration).
- We obtain $R(0)$ by a shooting method so that asymptotically the solution matches the de Sitter solution $R = R_1 = const.$ where R_1 is a critical point of the "potential" $V(R) = -Rf(R)/3 + \int^R f(x)dx$. That is, R_1 is a point where $dV(R)/dR = (2f - f_R R)/3$ vanishes.
- This potential is radically different from the scalar-field potential that arises under the STT map. Furthermore, $V(R)$ is as well defined as the function $f(R)$ itself.

- We used first the model $f(R) = R - \alpha R_* \ln(1 + R/R_*)$ (α, R_* are positive constants; R_* sets the scale) proposed by **Miranda et. al.** (PRL **102**, 221101, 2009). Warning: for this model f_R is not positive definite in general (only if $\alpha < 1 + R/R_*$) but $f_{RR} > 0$. Note also that $f(R)$ is only well defined for $0 < 1 + R/R_*$. A priori there is no guarantee that solutions for R exist satisfying such conditions.
- Those authors mapped the theory to the STT counterpart. However, unlike the Starobinsky model (see below), in this case the resulting scalar-field potential turns to be single-valued. This is why we take it in order to compare (calibrate) directly with our method.
- They did not find any singularity within the object.
- Under our approach we associate to this $f(R)$ the potential $V(R) = \frac{R_*^2}{6}\left\{(1+\tilde{R})\left[\tilde{R} + (6\alpha - 1)\right] - 2\alpha(3 + 2\tilde{R})\ln\left[1 + \tilde{R}\right]\right\}$, where $\tilde{R} = R/R_*$. For $\alpha = 1.2$ (the value that Miranda et al. assumed) this potential has several critical points

Some exact solutions (a small detour)



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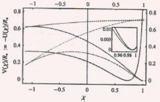
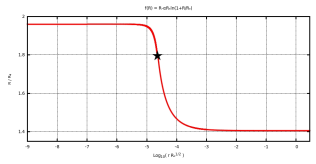
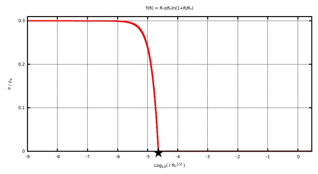
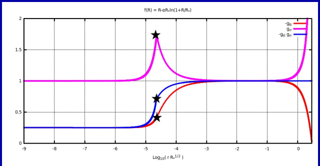


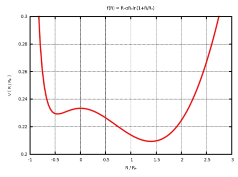
FIG. 1 (color online). $V(\chi)/R_* := -U(\chi)/R_*$ for different models: ours, with $\alpha = 2$ (blue solid line), Starobinski's for $\{n = 2, \lambda = 2\}$ [see [9]] (red dashed line), and Hu and Sawicki's for $\{n = 2, m^2 = 1, c_1/c_2 = 2\}$ [see [8]] (green dot-dashed line). The physically interesting region is $0 < \chi < 1$. For the multi-valued potentials only the lower lines are physical.

We have used our approach and found no singularities whatsoever in compact objects (like in Miranda et al.)

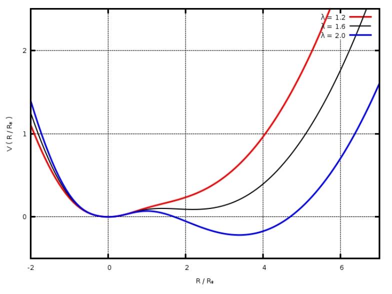




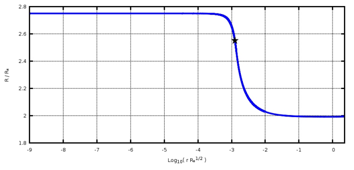


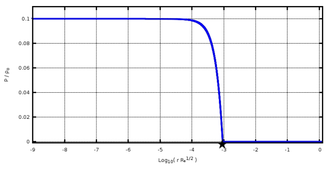


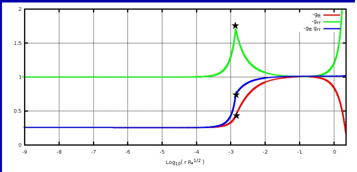
- We then used the Starobinsky model $f(R) = R - \lambda R_* \left\{1 - \left[1 + (R/R_*)^2\right]^{-\beta}\right\}$ with $\beta = 1$. The controversy about the existence of extended relativistic objects (or absence thereof) was originated using this model. As we saw, the STT approach gives rise to multivalued potentials.
- With our method, the potential is given by ($\tilde{R} = R/R_*$)
$V(R) = \frac{R_*^2}{3}\left\{\frac{\tilde{R}}{2}\left[\tilde{R} - 4\lambda - 2\lambda\left(1 + \tilde{R}^2\right)^{-1}\right] - 3\lambda \arctan\left(\tilde{R}\right)\right\}$.
- This potential has a rich structure depending on the value of λ.

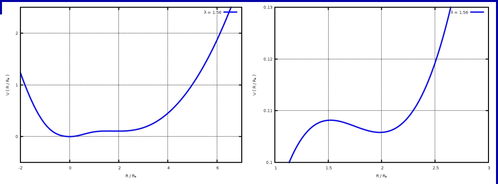


For the value $\lambda = 1.56$ we found the following solutions using a shooting method aiming to the local minimum. No singularities whatsoever where found.





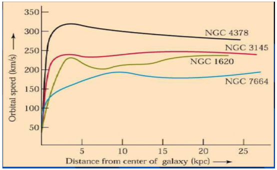




Discussion:
Although we have not find singularities in the spacetime generated by this compact objects, there is a caveat in the above construction:
In almost all $f(R)$ models the dimension parameters like R_*, which settles the scale $\sim D^{-2}$, are chosen such that $R_*/G_0 \sim \tilde{\Lambda} \sim 10^{-29} \mathrm{g\,cm}^{-3}$ ($\tilde{\Lambda} = \Lambda/G_0$). On the other hand, if one wants to build a realistic neutron star, the typical densities at the center are $\rho(0) \sim \rho_{\rm nuc} \sim 10^{14} \mathrm{g\,cm}^{-3}$. That is, there are around 43 orders of magnitude between the typical density within a neutron star and the average density of the Universe! This ratio between densities naturally appears in the equations since the parameters which define the specific $f(R)$ theory are of the order of $\tilde{\Lambda}$, while the appropriate dimension within neutron stars is $\rho_{\rm nuc}$. So in units of $\rho_{\rm nuc}$, the cosmological constant turns out to be ridiculously small, while in units of $\tilde{\Lambda}$, $\rho(r)$ and $p(r)$ turn out to be ridiculously large within the neutron star.

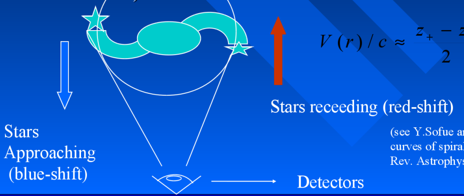
In other words, the scale of a neutron star is $\sim \mathrm{km}$ while the cosmological scales are $\sim 100\mathrm{Mpc}$. We, like the other authors, have not solved this technical problem...YET, and therefore have constructed "compact" objects which whether are realisitic but then $\tilde{\Lambda}$ is not. Or the opposite. In both cases the objects are compact and relativistic in the sense that $p \sim \rho$ and $G_0\mathcal{M}/\mathcal{R}$ is not far from $4/9$, where $\mathcal{M}$ is an ADM mass defined in asymptotically de Sitter spacetimes.

Later (1970's), observations in spiral galaxies showed that gas and stars revolved around the center of the galaxy in a very peculiar way: their tangential velocity $V(r) \approx const.$ for some $r > r_b \sim 5kpc$ instead of falling off as $V^2 \sim M_{vis}/r$, according to the Newtonian expectations. Here $M_{vis} \sim const. \sim 10^{11} M_\odot$ for $r > r_b$ stands for the total visible mass within the galactic core with radius r_b.



2) Empirical evidence: Since quite a long time (since time of Zwicky) there was evidence that astrophysical matter behaved in a unusual way (dynamics in clusters of galaxies). It seems that it was Zwicky himslef who coined the term “dark matter” to explain the dynamics of some of such systems.

More recently, observations in spiral galaxies showed that gas, and stars revolved around the center of the galaxy in a very peculiar way (the so called *Flat Rotation Curves of Galaxies*)



$$V(r)/c \approx \frac{z_+ - z_-}{2}$$

(see Y.Sofue and V.Rubin, ‘Rotation curves of spiral Galaxies “, Annu. Rev. Astrophys., 2001, vol.39, p.137)

In order to explain these curves, people started to introduce the idea that around the visible galactic matter there exists a **Dark Matter Halo** which is 10 times more massive than the luminous matter and which produce stronger gravity. Since according to Newtonian expectations $V^2/r \sim M(r)/r^2$, where now $M(r)$ accounts for the total mass including the Dark and visible Matter, one needs then that $M(r) \sim r$ grows linearly for $V(r) \approx const$. Moreover, since $M \sim \int \rho r^2 dr$, the dark matter contribution to the density should behave as $\rho \sim 1/r^2$. So one of the challenges of the Dark Matter supporters is to find a Universal self-gravitating dark matter model whose energy density behave this way.

Rotation curves in spiral galaxies: Assume that the spacetime generated by a galaxy is static and spherically symmetric.

$$ds^2 = -N^2(r)dt^2 + A^2(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (136)$$

Assume that gaz and stars in the galaxy's arms behave as test particles and revolve around the center in circular orbits on the galactic plane. Then the difference of blue shift and red shift from the matter approaching and receeding from us are given by

$$z_D = \frac{z_+ - z_-}{2} = \frac{1}{N}\left(1 - r\partial_r N/N\right)^{-1/2}\left(r\partial_r N/N\right)^{1/2} \quad (137)$$

Usually z_D is identified with the tangential velocity of the revoling matter. This result in theory independent. Usually $N(r)$ is given from the solution of the field equations of the underlying theory. For instance in GR one has $N = (1 - 2GM/r)^{1/2}$ $\approx (1 - GM/r)$. So at first order in GR/r one has $z_D \approx (GR/r)^{1/2}$. However, for $f(R)$ gravity the result might be different. Notably if $f(R)$ is given *a priori*. But one can follow a different strategy like in many Dark Matter models: one can fix $z_D = const$. then obtain $N(r)$ from Eq.(137) and from the field equations reconstruct $f(R)$. This approach has been followed by several authors. Nevertheless there is no guarantee that such a designer model will satisfy all the remaining observations nor that such a model will serve to reproduce all the rotation curves.