Black Holes' New Horizons
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Entropy of extremal black holes

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Outline

1. Introduction

Entropy is related to degrees of freedom. Matter entropy is related to the volume, e.g., Sackur-Tetrode entropy (1912), the entropy of a monatomic classical ideal gas adding quantum considerations $S = N\left(\ln\left(\frac{V}{N\lambda^3}\right) + \frac{5}{2}\right)$, with $\lambda = [3\pi/(mU/N)]^{1/2}$ the thermal wavelength ($c = 1$, $\hbar = 1$, $k_B = 1$).

Black hole entropy is in the area, the Bekenstein-Hawking entropy $S = \frac{1}{4}A$ in $A_{\rm p}$ units, ($G = 1$, $c = 1$, $\hbar = 1$, $k_B = 1$). Points to the ultimate degrees of freedom are in the area not volume. Works of 1970s.

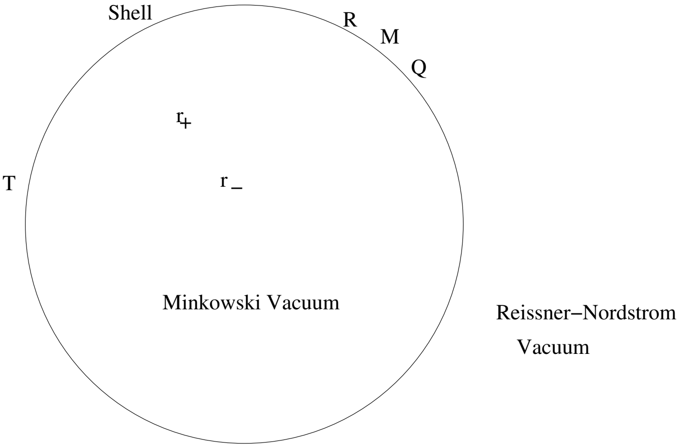
This is well established for nonextremal black holes: thermodynamics of black holes, Euclidean formulation and path integral approach to statistical mechanics.

Not so for extremal black holes. The Euclidean formulation shows that $S = 0$ due to trivial topology (Hawking, Horowitz, Ross 1995, Teitelboim 1995). On the other hand string theory formulation of extremal black holes shows $S = \frac{1}{4}A$ (Strominger, Vafa 1996). There is a problem here.

We use matter to study black hole entropy. Use the simplest form of matter: a shell. Amazingly, it reflects and gives a solution to the debate.

2. The dynamical and thermodynamic quantities

The study of charged thin shell involves three dynamical variables: the radius R of the shell, its rest mass M and its charge Q. For thermodynamics we also need T.



Find then p, Φ, and S from general relativity and thermodynamics.

2. The dynamical and thermodynamic quantities

Assuming that the shell has a well defined temperature, the first law of thermodynamics is

$$TdS = dM + pdA - \Phi dQ\,.$$

All thermodynamical quantities depend on A, M, and Q. It is useful to change to R, r_+, and r_-, the gravitational or horizon radius and the Cauchy radius,

$$r_+(R,M,Q) = m + \sqrt{m^2 - Q^2}\,,$$

$$r_-(R,M,Q) = m - \sqrt{m^2 - Q^2}\,,$$

where

$$m(R,M,Q) = M - \frac{GM^2}{2R} + \frac{Q^2}{2R}\,.$$

is the ADM mass. Define, k as

$$k(R,r_+,r_-) = \sqrt{\left(1 - \frac{r_+}{R}\right)\left(1 - \frac{r_-}{R}\right)}\,,$$

usually called the redshift function.

2. The dynamical and thermodynamic quantities

The quantities M and Q can be written in terms of (R, r_+, r_-),

$$M(R, r_+, r_-) = R(1-k)\,,$$

$$Q(R, r_+, r_-) = \sqrt{r_+ r_-}\,.$$

The area of the shell is

$$A(R, r_+, r_-) = 4\pi R^2\,,$$

and the gravitational area or horizon area is

$$A_+(R, r_+, r_-) = 4\pi r_+^2\,.$$

It will prove useful to keep the generic functional dependence.

In order for the electric charged shell to remain static, its tangential pressure must have a specific functional form, given by

$$p(R, r_+, r_-) = \frac{R^2(1-k)^2 - r_+ r_-}{16\pi R^3 k}\,.$$

2. The dynamical and thermodynamic quantities

The integrability conditions out of the first law of thermodynamics assert that the electric potential Φ

$$\Phi(R, r_+, r_-) = \frac{c(r_+, r_-) - \frac{1}{R}}{k} \sqrt{r_+ r_-}\,,$$

where $c(r_+, r_-)$ is an arbitrary function, which physically represents the electric potential of the shell multiplied by its charge, if it were located at infinity. Additionally, we need the non-extremal shell to have a well defined electric potential in the horizon limit. This leads to

$$c(r_+, r_-) = \frac{1}{r_+}\,,$$

and consequently

$$\Phi(R, r_+, r_-) = \sqrt{\frac{r_-}{r_+}} \sqrt{\frac{1 - \frac{r_+}{R}}{1 - \frac{r_-}{r_+}}}\,.$$

2. The dynamical and thermodynamic quantities

Assuming that the shell has a well defined temperature, the integrability conditions imposed from the first law of thermodynamics give

$$T(R, r_+, r_-) = \frac{T_0(r_+, r_-)}{k},$$

where T is the temperature at the shell and T_0 is the temperature seen from infinity. Now, we impose

$$T_0 = T_H = \frac{r_+ - r_-}{4\pi r_+^2},$$

where T_H is the Hawking temperature of an electrically charged black hole. So $T(R, r_+, r_-) = \frac{T_H(r_+, r_-)}{k}$, i.e.,

$$T(R, r_+, r_-) = \frac{r_+ - r_-}{4\pi r_+^2 \, k}.$$

3. Approach to the extremal horizon: The variables that define the three extremal horizon limits

To study independently the limit of an extremal shell and the limit of a shell being taken to its gravitational radius, it will prove fruitful to define the variables ε and δ through the equations

$$1-\frac{r_-}{R}=\delta^2\,,$$

$$1-\frac{r_+}{R}=\varepsilon^2\,.$$

It is clearly seen that these the variables ε and δ are the good ones to take the extremal limit. There are however different extremal limits depending on how ε and δ are taken to zero.

Using the equations above we immediately get that the redshift function is

$$k(R,\varepsilon,\delta)=\varepsilon\delta\,.$$

In these variables it depends on ε and δ.

4. Geometry: The three extremal horizon limits

There are three physically relevant, limits. Let us see the geometry.

Case 1. In this case we do $r_+ \neq r_-$ and $R \to r_+$, i.e.,

$$\delta = O(1)\,, \quad \varepsilon \to 0\,.$$

Thus there is the horizon limit, but there is no extremal limit, the shell remains nonextremal during the whole process. After the calculations are done and we have an expression for the entropy we take the $\delta \to 0$ limit.

Case 2. In this case we do $R \to r_+$ and $r_+ \to r_-$, i.e.,

$$\delta = \frac{\varepsilon}{\lambda}\,, \quad \varepsilon \to 0\,,$$

where the new parameter λ remains constant and it must satisfy $\lambda < 1$ due to $r_+ \geq r_-$. It means that simultaneously $R \to r_+$ and $r_+ \to r_-$ in such a way that $\delta \sim \varepsilon$. In other words, the horizon limit is accompanied with the extremal one.

Case 3. In this case we do $r_+ = r_-$ and $R \to r_+$, i.e.,

$$\delta = \varepsilon\,, \quad \varepsilon \to 0\,.$$

This corresponds to the extremal shell.

5. Mass and electric charge: The three extremal horizon limits

Inverting gives

$$M(R,\varepsilon,\delta) = R(1-\varepsilon\delta)\,,$$

$$Q(R,\varepsilon,\delta) = R\sqrt{(1-\varepsilon^2)(1-\delta^2)}\,.$$

We can then study the three cases.

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, we get

$$M(r_+,\varepsilon,\delta) = r_+\,, \quad Q(r_+,\varepsilon,\delta) = r_+\sqrt{1-\delta^2}\,.$$

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$ and $\varepsilon \to 0$

$$M(r_+,\varepsilon,\delta) = r_+\,, \quad Q(r_+,\varepsilon,\delta) = r_+\,.$$

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$ we get

$$M(r_+,\varepsilon,\delta) = r_+\,, \quad Q(r_+,\varepsilon,\delta) = r_+\,.$$

6. Pressure, electric potential and temperature: The three extremal horizon limits

The tangential pressure in terms of the variables ε and δ is

$$p(R,\varepsilon,\delta) = \frac{1}{16\pi R}\frac{(\delta-\varepsilon)^2}{\delta\varepsilon}\,.$$

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, we get

$$p(r_+,\varepsilon,\delta) = \frac{\delta}{16\pi r_+ \varepsilon} \sim \frac{1}{\varepsilon}\,.$$

So, the pressure is divergent in this case as $1/\varepsilon$.

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$ and $\varepsilon \to 0$

$$p(r_+,\varepsilon,\delta) = \frac{1}{16\pi r_+}\frac{(1-\lambda)^2}{\lambda}\,,$$

The pressure is finite but nonzero in this horizon limit for the extremal shell.

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$ it is seen from the equation above

$$p(r_+,\varepsilon,\delta) = 0\,.$$

The result $p = 0$ holds in fact at any radius, including the horizon limit.

6. Pressure, electric potential and temperature: The three extremal horizon limits

In terms of ε and δ the electric potential is

$$\Phi(R, \varepsilon, \delta) = \sqrt{\frac{1-\delta^2}{1-\varepsilon^2}} \frac{\varepsilon}{\delta}\,.$$

It is then possible to analyze the three limiting case.

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, we get

$$\Phi(r_+, \varepsilon, \delta) = 0\,.$$

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$, with λ kept fixed and $\varepsilon \to 0$,

$$\Phi(r_+, \varepsilon, \delta) = \lambda\,, \quad \lambda < 1\,.$$

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$ it gives $\Phi(r_+, \varepsilon, \delta) = 1$. Actually, the integrability condition for this case gives a more general result,

$$\Phi(r_+, \varepsilon, \delta) \leq 1\,.$$

The extremal shell from the very beginning differs from what is obtained by the extremal limit from the nonextremal state.

6. Pressure, electric potential and temperature: The three extremal horizon limits

In terms of ε and δ the temperature is $T_H(R,\varepsilon,\delta) = \frac{\delta^2-\varepsilon^2}{4\pi R(1-\varepsilon^2)^2}$ and so the local temperature on the shell is thus

$$T(R,\varepsilon,\delta) = \frac{T_H}{k} = \frac{\delta^2-\varepsilon^2}{4\pi R\delta\varepsilon(1-\varepsilon^2)^2}\,.$$

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, we get

$$T(r_+,\varepsilon,\delta) = \frac{\delta}{4\pi r_+\varepsilon} \sim \frac{1}{\varepsilon}\,.$$

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$ and $\varepsilon \to 0$,

$$T(r_+,\varepsilon,\delta) = \frac{1-\lambda^2}{4\pi r_+\lambda}\,.$$

It remains finite and nonzero. Note a simple formula in this limit $\frac{p}{T} = \frac{1}{4}\frac{1-\lambda}{1+\lambda}$.

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$, one can have $T_0 = T_H \to 0$ but T remains finite.

7. Entropy: The three extremal horizon limits

Case 1. For $r_+ \neq r_-$ and as $R \to r_+$, i.e., for $\delta = O(1)$ and as $\varepsilon \to 0$, we express the first law of thermodynamics the variables (R, ε, δ), $dS(R, \varepsilon, \delta) = 2\pi R \left(1 - \varepsilon^2\right)^2 dR - 4\pi R^2 \varepsilon \left(1 - \varepsilon^2\right) d\varepsilon$. This integrates to $S(R, \varepsilon, \delta) = \pi R^2 \left(1 - \varepsilon^2\right)^2$. So, in this limit,

$$S(r_+) = \frac{A_+}{4}\,,$$

where A_+ is the horizon area. It is the Bekenstein-Hawking entropy. We can now take the extremal limit $\delta \to 0$ and obtain that the entropy of an extremal charged black hole is $S(r_+) = \frac{A_+}{4}$, the Bekenstein-Hawking entropy.

Case 2. For $R \to r_+$ and $r_+ \to r_-$, i.e., for $\delta = \frac{\varepsilon}{\lambda}$, with λ kept fixed and $\varepsilon \to 0$ we obtain

$$S(r_+) = \frac{A_+}{4}\,,$$

$S(r_+) = \frac{A_+}{4}$. In this limit the entropy of an extremal black hole obtained through a non-extremal shell by means of the limiting transition under discussion is also equal to the Bekenstein-Hawking entropy.

7. Entropy: The three extremal horizon limits

Case 3. For $r_+ = r_-$ and as $R \to r_+$, i.e., for $\delta = \varepsilon$ and $\varepsilon \to 0$, the first law is $TdS = dM - \Phi dQ$. Putting $\beta = 1/T$, and $M = Q = r_+$ get $dS = \beta\,(1-\Phi)\,dr_+$. Then, integrability condition gives $\beta\,(1-\Phi) = s(r_+)$. So

$$S(r_+) = \text{a physically well behaved function of } A_+\,.$$

A result different from 0 or $A_+/4$, perhaps $0 \le S \le A_+/4$. It suggests that the entropy of extremal black holes may depend on their prior history.

A table briefly summarizes our results. It is implied that in all three cases the horizon limit is taken.

Case	p	Φ	T	S	Contribution to 1st law
1	diverges ε^{-1}	0	ε^{-1}	$A_+/4$	pressure
2	nonzero	< 1	nonzero	$A_+/4$	energy, press. and electric.
3	0	≤ 1	nonzero	$f(A_+)$	mass and electricity

9. References

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