

# Multivariable Operator Theory and Function Theory in Several Variables

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## 1 Overview of the Field

For 70 years there have been such strong links between operator theory and functions of a single complex variable that the two subjects were effectively indivisible: one could not study one without the other. This has not been the case to the same extent with several variables; the several complex variable community has traditionally been more distinct from the multivariable operator theory community.

However, recent progress is leading to significant further integration. Operator theoretic methods have shed light on function theory on the bidisk and the symmetrized bidisk, and to understanding the Drury-Arveson space and other Besov-like spaces of holomorphic functions on the ball that have a desirable property (the complete Pick property). Conversely, hard estimates from function theory of several variables have been used to prove that Andô's inequality holds for any  $d$ -tuple of 3-by-3 matrices, and to resolve questions about which subvarieties of the ball satisfy an Andô-like inequality.

Furthermore, advances in non-commutative function theory have led to questions about holomorphic functions of several complex matrices, which seem to require both complex analysis and operator theory to resolve. This is an exciting area, with links to non-commutative probability as well, experiencing explosive growth.

## 2 Recent Developments and Open Problems

For brevity, we shall just mention two of the several areas that were covered at the Conference.

**NON-COMMUTATIVE FUNCTION THEORY AND ITS APPLICATIONS:** Noncommutative function theory originated in the work of Joseph Taylor in the 1970's. His idea was that, just as a holomorphic function can be thought of, in a variety of ways, as a generalized polynomial, so a non-commutative function would be a sort of generalized non-commutative (or free) polynomial. The natural arguments, in his view, were  $d$ -tuples of  $n$ -by- $n$  matrices, but where the statements should somehow be independent of  $n$ . His ideas laid dormant for three decades, until early this century when there was a surge in interest from various quarters, including control theory, real algebraic geometry, detailed analysis of the Fock space, and a search for the free complex analysis to mirror the free real analysis of free probability.

The theory has not only been successful on its own terms but has also had applications to the commutative theory, such as the work of Jury and Martin that successfully resolved some outstanding questions about the

Drury-Arveson space that were not initially formulated in noncommutative terms [4]. Moreover, the theory has provided a new perspective on previously studied objects, such as the free analytic Toeplitz algebra, which was identified with the algebra of bounded non-commutative functions on the row ball by Salomon, Shalit and Shamovich [5].

**MULTIPLIER SPACES AND THEIR PROPERTIES:** The algebra  $H^\infty$  of bounded analytic functions on the unit disk has been the focus of much attention in the fields of operator theory and complex analysis for many decades. The crucial connection to operator theory is the observation that  $H^\infty$  is the multiplier algebra of the Hardy space  $H^2$ , a Hilbert function space.

A particularly tractable class are multiplier algebras of complete Pick spaces. In the last two decades, there has been significant progress on questions surrounding these algebras and related multiplier spaces by a number of groups. Examples include corona theorems (Costea, Sawyer and Wick [2]), the study of Carleson measures, the characterization of interpolating sequences (Aleman, Hartz, McCarthy and Richter [1]), the study of extreme points and de Branges-Rovnyak spaces (Jury and Martin [3]), Beurling theorems, and the isomorphism problem for multiplier algebras (Davidson, Hartz, McCarthy, Ramsey, Shalit [6]). Recently, the subject has experienced a new influx of ideas from non-commutative function theory, for instance in the work of Jury and Martin.

### 3 Presentation Highlights and Outcome of the Meeting

We adopted an unusual format for the presentations. Speakers were asked to talk not on their own work, but on the work of others, with the exception of a few junior participants. This turned out to be very successful—the talks gave broad overviews of advances in the field in the last couple of years, and were of extremely high quality. This in turn led to compelling and accessible presentations of some open questions, which is likely to stimulate further research.

On average, the workshop was attended by approximately 50 participants. One of the participants, Orr Shalit, recorded his positive experience with our unusual format on his personal blog; see [7].

### References

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