Parabolically induced representations of G_2 distinguished by SO_4

ANTD, BIRS

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Let G be a connected reductive group over a p-adic field F of characteristic zero ($\mathbb{Q}_p$ or finite extension of $\mathbb{Q}_p$), and let $R(G)$ be the category of complex representations $(\pi; V)$ such that $\{g \in G : \pi(g)v = v\}$ is open in G for all $v \in V$.

Example
$G = \mathrm{GL}_n$, $G = \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}$, $G = G_2$

Definition (Parabolically induced representation)
$P = M \ltimes U, (\sigma, V)$ a smooth representation of M.

$$\mathrm{Ind}_P^G(\sigma) := \{f : G \to V, f(pg) = (\delta_P^{1/2}\sigma)(p)f(g), p \in P, g \in G\}$$

G acts on these functions by right-translation: $(R(g)f)(x) = f(xg), g, x \in G$

Distinguished representations

Let G be a reductive group over a p-adic field F, and H a closed subgroup of G. Take a smooth complex-valued representation π.

Question: What are the representations π of G and characters χ of H such that $\mathrm{Hom}_H(\pi|_H, \chi) \neq 0$?

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Example: $\chi = \mathbf{1}_H$.

Case of interest: θ an involution G, $H = G^\theta$.

Why bother?:

Local motivation:

$$L^2(H\backslash G) = \int_{\hat{G}}^{\oplus} \pi \mu(\pi)$$

This is a unitary representation of G, and the Plancherel measure μ is supported on the class of H-distinguished representations.

Motivations

Why bother?:

Global motivation:

Let ϕ be an automorphic form on $G(\mathbb{A}_F)$, with F a number field. Then we are interested in understanding if

$$\int_{ZH(F)\backslash H(\mathbb{A}_F)} \phi(h)\chi(h)dh \neq 0$$

Relative Trace formula

Baby case: $GL_2 = GL_2(F)$, and its torus $T = T(F)$

$$B = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \ltimes \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad a, d \in F^\times, b \in F$$

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Take $\ell \in Hom_T(\mathrm{Ind}_B^{GL_2}(\chi), \mathbf{1})$. Then $\ell(R(z).f) = \ell(\chi(z).f) = \chi(z)\ell(f) = \ell(f)$, so we need

$$\chi|_Z = \mathbf{1}$$

Necessary condition: The representation $\mathrm{Ind}_B^{GL_2}(\chi)$ has a trivial central character.

Sufficient condition

We need to look at the double coset $B\backslash GL_2/T$. Using Bruhat decomposition, we find:

$$GL_2 = B \sqcup B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sqcup B \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} T$$

$$B\backslash GL_2/T = \{B = \{g_{21} = 0\}\,; Bw = \{g_{22} = 0\}\,; B\eta T = \{g_{21}g_{22} \neq 0\}\}$$

Since $B\eta T$ is open, we can look at the T-invariant space

$$V = \left\{ f \in \mathrm{Ind}_B^{GL_2}(\chi) | \mathrm{Supp}(f) \in B\eta T \right\}$$

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$$V = \left\{ f \in \mathrm{Ind}_B^{GL_2}(\chi) | \mathrm{Supp}(f) \in B\eta T \right\}$$

We then get the filtration: $0 \subseteq V \subseteq \mathrm{Ind}_B^{GL_2}(\chi)|_T$. If $\ell \in \mathrm{Hom}_T(\mathrm{Ind}_B^{GL_2}(\chi), 1)$ then either $\ell|_V \neq 0$ (in which case, $\mathrm{Hom}_T(V, 1) \neq 0$) or ℓ is a non-zero element of $\mathrm{Hom}_T(\mathrm{Ind}_B^{GL_2}(\chi)/V, 1)$.

Method: The Geometric Lemma

Remark: This method only allows to study $\pi \cong \mathrm{Ind}_P^G(\sigma)$

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$V_i = \left\{f \in \mathrm{Ind}_P^G(\sigma) : \mathrm{Supp}(f) \subseteq \mathcal{O}_i\right\}$ then $0 \subseteq V_0 \subseteq V_1 \subseteq \ldots \subseteq V_N = \mathrm{Ind}_P^G(\sigma)|_H$ is a filtration of $\mathrm{Ind}_P^G(\sigma)|_H$

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$V_i = \left\{f \in \mathrm{Ind}_P^G(\sigma) : \mathrm{Supp}(f) \subseteq \mathcal{O}_i\right\}$ then $0 \subseteq V_0 \subseteq V_1 \subseteq \ldots \subseteq V_N = \mathrm{Ind}_P^G(\sigma)|_H$ is a filtration of $\mathrm{Ind}_P^G(\sigma)|_H$

If $\mathrm{Ind}_P^G(\sigma)$ is H-distinguished, then there exists i such that $\mathrm{Hom}_H(V_i/V_{i-1}; \mathbf{1}) \neq 0$. Indeed, if $\ell \in \mathrm{Hom}_H(\mathrm{Ind}_P^G(\sigma);\ \mathbf{1}) \neq 0$ then there exists i minimal such that $\ell|_{V_i} \neq 0$ defines a non-zero element of $\mathrm{Hom}_H(V_i/V_{i-1}; \mathbf{1})$.

Lemma

$$\mathrm{Hom}_H(V_i/V_{i-1}; \boldsymbol{1}) = \mathrm{Hom}_{P_i}(\delta_{P_i}^{-1}\delta_P^{1/2}\sigma, \boldsymbol{1})$$

where $P_i = \eta_i H \eta_i^{-1} \cap P$.

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where $P_i = \eta_i H \eta_i^{-1} \cap P$.

Set $x_i = \eta_i\theta(\eta_i)^{-1}$, $\theta_x(g) = x\theta(g)x^{-1}$ and $M_x = (M \cap \theta_x(M))^{\theta_x}$.

Proposition (Closed orbit, Offen, 2017)
Let $P = M \ltimes U$ *be a standard parabolic subgroup of* G *and* σ *a smooth representation of* M*. Suppose that* x *is* M*-admissible, and* $\theta_x(P) = P$*. If* σ *is* $(M_x, \delta_{P_x}\delta_P^{-1/2}\chi^{\eta^{-1}})$*-distinguished then* $\mathrm{Ind}_P^G(\sigma)$ *is* (H,χ)*-distinguished.*

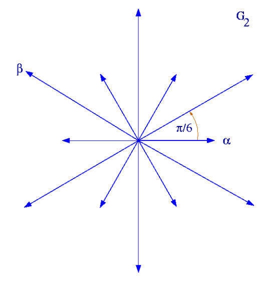
What is G_2?

$k, \mathbb{K}, \mathbb{B}, \mathcal{C}$ Hurwitz algebras of dimension 1,2,4,8 (quaternions, octonions over k).
G_2 is the group of automorphisms of the Cayley algebra. We embed G_2 into GL_8 using the action of the root subgroups/the torus of G_2 on the octonions.

The case of $(\mathrm{G}_2, \mathrm{SO}_4)$

In this talk, we will denote $G_2 = \mathrm{G}_2(F)$, $\mathrm{SO}_4 = \mathrm{SO}_4(F)$...etc

Our goal is to determine for which σ, and χ_{SO_4}, $\mathrm{Hom}_{\mathrm{SO}_4}(I_P^{\mathrm{G}_2}(\sigma), \chi_{\mathrm{SO}_4}) \neq 0$.



Lemma **The quotient $\mathrm{G}_2/\mathrm{SO}_4$ is a symmetric space**

Proposition (Closed orbit, Offen, 2017)
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Our main results are the following:

Theorem (Closed orbit, D, 2022)
Let χ be a character of $\mathrm{SO}_4(F)$. It is a quadratic character of $F^\times$. It can be seen as a character of GL_2 (those are given by $\chi \circ \det$ for a quasi-character χ of $F^\times$). Let P_β (resp. P_α) denote the maximal parabolic corresponding to the root β (resp. α). The parabolic induced representations of G_2 which are (SO_4, χ)-distinguished include the following representations:

- *The induction from P_β to G_2 of the reducible principal series $I(\chi\delta_{P_\beta}^{1/2}|.|^{-1/2} \otimes |.|)$ of GL_2.*
- *The induction from P_α to G_2 of the reducible principal series $I(\chi\delta_{P_\alpha}^{1/2}|.|^{1/2} \otimes \chi\delta_{P_\alpha}^{1/2}|.|^{-1/2})$of GL_2.*
- *The induced representation $I_{P_\beta}^{G_2}((\chi \circ \det)\delta_{P_\beta}^{1/2})$*
- *The induced representation $I_{P_\alpha}^{G_2}((\chi \circ \det)\delta_{P_\alpha}^{1/2})$.*

A more structural approach

Joint work with Nadir Matringe.

Definition

A quaternion algebra $D = D_{\alpha,\beta}$ over a field F is a 4-dimensional vector space over F, with basis $\{1, i, j, k\}$, given the structure of an algebra with the multiplication rules

$$i^2 = \alpha, \;\; j^2 = \beta$$

and $ij = -ji = k$ for some α and β in $F^\times$.

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Fact 1: A split quaternion algebra $\cong M_2(F)$. We use the **Cayley-Dickson construction** of the split octonions: $\mathbb{O}_p = M_2(F) \oplus M_2(F)$, equipped with the norm $n_{\mathbb{O}_p}((x, y)) = \det(x) - \det(y)$.

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Fact 1: A split quaternion algebra $\cong M_2(F)$. We use the **Cayley-Dickson construction** of the split octonions: $\mathbb{O}_p = M_2(F) \oplus M_2(F)$, equipped with the norm $n_{\mathbb{O}_p}((x, y)) = \det(x) - \det(y)$. SO_4 is the subgroup of automorphisms of the octonions that fix a **split** quaternionic subalgebra!

SO_4-orbits of the nil-subalgebras of G_2

Our question today: describe structurally $P_\beta \backslash G_2/\mathrm{SO}_4$ **and** $P_\alpha \backslash G_2/\mathrm{SO}_4$

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Definition (Nil-subalgebra)
A nil-subalgebra is a subspace of $\mathbb{O}_p$ consisting of **trace zero elements with trivial multiplication** (the product of any two elements is zero).

The quotient $P_\beta \backslash G_2$ (resp. $P_\alpha \backslash G_2$) correspond to the set of nil-subalgebras of dimension 1 (resp. dim 2) of the split octonions $\mathbb{O}_p$.

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Fact 2: SO_4 is the centralizer of the involution ϵ in $\mathbb{O}_p$: $\epsilon(a, b) = (a, -b)$.

Fact 3: $\mathrm{SO}_4 = \mathrm{SL}_{2,s} \times \mathrm{SL}_{2,l}/\Delta\mu_2 = PGL2 \times SL_2$ and acts on $M_2(F) \oplus M_2(F)$ by $(g, h)(x, y) = (gxg^{-1}, hgyg^{-1})$.

Lemma

Any element in a nil-subalgebra of the split octonions has the form: $\gamma(M_0, M)$ *for* γ *in* F*, with* $M_0^2 = -\det(M)I_2$ *and* $Tr(M_0)M = 0$.

Proof.
Just an application of the 2-nilpotency of such element and using the multiplication law of the octonions

$$(a + \ell b)(c + \ell d) = (ac + \lambda \bar{d} b) + \ell(da + b\bar{c})$$

□

Lemma

There are $|F^\times/(F^\times)^2| + 4$ *orbits for the action of* SO_4 *on* $P_\beta \backslash G_2$.

Proof:

Using the description of the nil-subalgebras of dimension 1 above, let us distinguish three cases where the nil-subalgebra is generated by: (M_0, M) with $M = 0$ (1), (M_0, M) with $M_0 = 0$ (2), and (M_0, M) with both M_0 and M non-zeros (3).

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(1) Since $M = 0$, we have only the condition $M_0^2 = 0$ (i.e $Tr(M_0)$ is not necessarily 0). Remembering that the PGL_2-conjugacy classes of $M_2(F)$ are parametrized by Jordan types, and using the fact that M_0 also needs to be **2-nilpotent**, there is only one element: $M_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(2) We get the equation, $\det(M) = 0$, we have two Jordan types, but as we let SL_2 acts by translation on the left, they are in the same orbit, so once more we are left with the nilpotent conjugacy classes in the set $M_2(F)$ under the action of SL_2. The nil-lines (1), and (2) constitute the two closed parabolic orbits.

(3) Finally, **let us assume both M_0 and M are non-zero.**
First, we notice that $k(M_0, M)$ with $k \in F$ with both matrices of rank 1, and $k'(M_0', M'), \quad k' \in F$ with M_0' and M' of rank 2 can not be in the same SO_4-orbit.
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Actually the first case gives us already two additional orbits 3a).
Let us consider now the second case 3b). We the equation
$M_0^2 = -\det(M) I_2$.

The minimal polynomial for M_0 is $P(x) = x^2 + \det(M)I$, and it is also the characteristic polynomial, since $\det(M) = \det(M_0)$. As the characteristic and the minimal polynomials are equal, there is only one conjugacy class of matrices M_0 verifying $P(M_0) = 0$.

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In particular, multiplying such a matrix by a scalar t would give us the equation $t^2(M_0)^2 = -\lambda I_2$, so the square classes in F (they are four of them) **parametrize this set**.

Because the four orbits are characterized by the equations $\det(M) \neq 0, \det(M_0) \neq 0$, these parabolic orbits are open.

List of the orbits of nil-lines

$$\begin{cases} \left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \text{ and } \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0\right) & \text{closed orbits} \\ \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \text{ and } \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) & \text{rank one orbits} \\ \left(\begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -t \end{pmatrix}\right) & \text{open orbits} \end{cases}$$

Lemma (Tentative lemma)
There are $|F^\times/(F^\times)^2| + 2$ *orbits for the action of* SO_4 *on* $P_\alpha \backslash G_2$*.*

Proof:

We look for pairs of nil-lines (N_1, N_2) so that $N_1.N_2 = 0$, and let SO_4 acts diagonally on such pairs. Further, we need to consider the nil-subalgebras, i.e Vect(N_1, N_2). Let $\mathcal{N}$ be such a 2-dimensional nil-subalgebra, if $\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$ belongs to $\mathcal{N}$, then using the nilpotency equations, and some simplification using the SO_4-action, we get

$$\mathcal{N} := F.\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \oplus F.\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right).$$

If $\left(\begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -t \end{pmatrix} \right)$ belongs to $\mathcal{N}$, let $N_2 := \left(u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, v = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right)$.
Using the nilpotency equation, we get

$$(u, v) = \left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} b/t & a \\ a & 0 \end{pmatrix} \right)$$

Then deal with the case $a = 0$ and $a \neq 0$ separately. In the first, get one orbit, and in the second, get again $|F^\times/(F^\times)^2|$ of them (open).

Thank you for your attention!