Sums of proper divisors with missing digits

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Let $s(n)$ denote the sum of proper divisors of n, i.e.,

$$s(n) := \sum_{\substack{d \mid n \\ d < n}} d = \sigma(n) - n,$$

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Abundant numbers: 12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, 102, ...

Deficient numbers: 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, ...

Perfect numbers: 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, ...

The function $s(n)$

If $\mathcal{S}$ is a subset of the natural numbers, then the *asymptotic density* of $\mathcal{S}$ is given by

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : n \in \mathcal{S}\},$$

if the limit exists.

Theorem (Davenport, 1933)

For every $u > 0$, let $\mathcal{D}(u) = \{n \in \mathbb{N} : s(n)/n \leq u\}$. Then $\mathcal{D}(u)$ has an asymptotic density for every u. Let $D(u)$ denote this density, then $D(u)$ is continuous everywhere and $\lim_{u\to\infty} D(u) = 1$.

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- The set of deficient numbers has asymptotic density $D(1)$. (around 75.23% (Déléglise, 1998), Kobayashi (2014))
- The set of abundant numbers has asymptotic density $1 - D(1)$. (around 24.76%)
- The set of perfect numbers has asymptotic density 0.

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Then $\#(\mathcal{A} \cap [1,x]) \ll x(\log\log x)/\log x$. So, $\mathcal{A}$ has asymptotic density zero.

Note that $s(pq) = p + q + 1$.

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The number of even integers less than x which are not the sum of two primes is at most $x^{1-\delta}$ for some $\delta > 0$.
(Montgomery and Vaughan (1975) improving on works of Chukadov (1937), van der Corput (1937) and Estermann (1938). Pintz (2018): $\delta = 0.28$ works).

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So, $s(\mathcal{A}) = \{p + q + 1 : \ p, q \text{ distinct primes}\}$ contains almost all odd numbers so that the asymptotic density of $s(\mathcal{A})$ is $1/2$.

The preimages of $s(n)$-EGPS Conjecture

Conjecture (Erdős, Granville, Pomerance, Spiro, 1990)

Let $\mathcal{A}$ be a set with asymptotic density 0. Then $s^{-1}(\mathcal{A})$ also has asymptotic density 0.

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Except for the 2017 result, the proof all of the above heavily depend on the arithmetic properties of the sets under consideration.

(Ellipsephic) Integers with missing (restricted) digits

Let $g \in \mathbb{N}$, $g \geq 3$. Consider the base g expansion of $n \in \mathbb{N}$,

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For a proper subset $\mathcal{D} \subsetneq \{0, \ldots, g-1\}$, let

$$\mathcal{W}_{\mathcal{D}} := \left\{ n : n = \sum_{j=0}^{N} \varepsilon_j(n) g^j, \varepsilon_j(n) \in \mathcal{D}, \; N \in \mathbb{N} \right\}$$

as the set of integers whose the digits are restricted in the set $\mathcal{D}$.

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The elements of $\mathcal{W}_{\mathcal{D}}$ are called *integers with missing digits* (or *integers with restricted digits*). Mauduit referred to these numbers as *ellipsephic integers* (combining two Greek words "ellipsis" = missing and "psiphic" = digit).

Let

$$\mathcal{W}_{\mathcal{D}}(x) := \mathcal{W}_{\mathcal{D}} \cap [1, x].$$

If $0 \in \mathcal{D}$, we have

$$\#\mathcal{W}_{\mathcal{D}}\left(g^{N} - 1\right) = |\mathcal{D}|^{N},$$

and if $0 \notin \mathcal{D}$, then

$$\#\mathcal{W}_{\mathcal{D}}(g^{N} - 1) = \sum_{\ell=0}^{N-1} |\mathcal{D}|^{\ell+1} = |\mathcal{D}| \frac{(|\mathcal{D}|^{N} - 1)}{(|\mathcal{D}| - 1)}$$

.

As $\#\mathcal{D} \leq g - 1$, the set $\mathcal{W}_{\mathcal{D}}$ has asymptotic density zero.

Integers with missing digits-arithmetic progressions

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For $q \in \mathbb{N}$ and $a \in \mathbb{Z}$, let $\mathcal{W}_\mathcal{D}(x, a, q) = \{n \leq x : \ n \equiv a \bmod q\}$.

Theorem (Erdős, Mauduit, Sárközy, 1998)

Let $\mathcal{D} = \{d_1, d_2, \ldots, d_t\}$, *with*

$$d_1 = 0 \in \mathcal{D} \textit{ and } \gcd(d_2, \ldots, d_t) = 1.$$

Then there exist constants $c_1 := c_1(g, t) > 0$ *and* $c_2 := c_2(g, t) > 0$ *such that*

$$\left| \#\mathcal{W}_\mathcal{D}(x, a, q) - \frac{\#\mathcal{W}_\mathcal{D}(x)}{q} \right| = O\left(\frac{\#\mathcal{W}_\mathcal{D}(x)}{q} \exp\left(-c_1 \frac{\log x}{\log q} \right) \right)$$

for all $a \in \mathbb{Z}$ *and* $1 < q \leq \exp(c_2\sqrt{\log x})$ *such that* $\gcd(q, g(g-1)) = 1$.

The range of q in this result was improved by Konyagin in 2001 and by Col in 2009.

On averaging over the modulus, the results can be extended to larger q .

Dartyge and Mauduit (2001) and independently Konyagin (2001) proved that there exists an $\alpha := \alpha(g, \mathcal{D})$ such that

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Banks and Shparlinski (2004) studied the average values of the Euler φ-function and the sum-of-divisors function, σ on $\mathcal{W}_{\mathcal{D}}$ among many other arithmetic properties of $\mathcal{W}_{\mathcal{D}}$.

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The problem of the existence of infinitely many primes with missing digits has been solved recently.

Theorem (Maynard, 2019)

Let $a_0 \in \{0, ..., 9\}$. The number of primes $p \leq x$ with no digit a_0 in their base 10 expansions is

$$\asymp \frac{x^{\frac{\log 9}{\log 10}}}{\log x}.$$

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Maynard also provided a condition to determine whether there are finitely or infinitely many n such that $P(n) \in \mathcal{W}_{\mathcal{D}}$, for any given non-constant polynomial $P \in \mathbb{Z}[X]$, large enough base g, and $\mathcal{D} = \{0, \ldots, g-1\} \setminus \{a_0\}$.

Integers with missing digits and $s(n)$

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Theorem (B., Cesana, Dartyge, Dombrowsky, and Thompson, 2023)

Fix $g \geq 3$, $\gamma \in (0,1)$, and a nonempty set $\mathcal{D} \subsetneq \{0, 1, \ldots, g-1\}$. For all sufficiently large x, the number of $n \leq x$ for which $s(n)$ has all of its digits in base g restricted to digits in $\mathcal{D}$ is $O(x \exp(-(\log\log x)^\gamma))$. That is,

$$\#\left(s^{-1}\left(\mathcal{W}_{\mathcal{D}}\right)\right)(x) \ll x \exp(-(\log\log x)^\gamma).$$

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Since $s(p) = 1$ for all primes p, whenever the set $\mathcal{D}$ contains 1, it follows that the size of the preimage set of $\mathcal{D}$ is $\gg \frac{x}{\log x}$ as $x \to \infty$. So, the upper bound in the theorem is essentially optimal in the sense that the constant $\gamma \in (0,1)$ could not be replaced by a constant strictly greater than 1.

A Key Lemma

Main ingredient: an upper bound for the number of positive integers $n \leq x$ such $g^k \nmid \sigma(n)$ when g^k is a large integer.

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Lemma (Watson (1935), Pomerance (1977))

Let $x \geq 3$ and q be a positive integer. The number of $n \leq x$ for which $q \nmid \sigma(n)$ is $O\left(\frac{x}{(\log x)^{1/\varphi(q)}}\right)$, uniformly in q.

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We can improve the upper bound, losing uniformity.

Lemma (BCDDT, 2023)

Let $g \geq 3$ be a given integer. Let $\gamma, \delta \in (0,1)$ and $A > 0$ also be given. Then for integers $k \in [\log\log\log x, A(\log\log x)^{\gamma}]$, we have

$$\sum_{\substack{n \leq x \\ g^k \nmid \sigma(n)}} 1 \ll x \exp\left(-(\log\log x)^{\delta}\right),$$

where the constant implied by the $\ll$ notation depends on the choices of g, A, γ, δ.

Proof that the lemma implies the theorem

Set $g \in \mathbb{N}$, $g \geq 3$, $\mathcal{D} \subsetneq \{0, 1, \ldots, g-1\}$ nonempty, x sufficiently large, and $0 < \gamma < 1$. For a positive integer

$$k \in \left[\frac{(\log \log x)^\gamma}{\log(g/|\mathcal{D}|)}, 2\frac{(\log \log x)^\gamma}{\log(g/|\mathcal{D}|)} \right],$$

we write

$$\# \left(s^{-1}(\mathcal{W}_\mathcal{D}) \right)(x) = \sum_{\substack{n \in \left(s^{-1}(\mathcal{W}_\mathcal{D})\right)(x) \\ \sigma(n) \equiv 0 \bmod g^k}} 1 + \sum_{\substack{n \in \left(s^{-1}(\mathcal{W}_\mathcal{D})\right)(x) \\ \sigma(n) \not\equiv 0 \bmod g^k}} 1 =: S_1 + S_2.$$

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For S_2, we use our lemma and obtain

$$S_2 = \sum_{\substack{n \leq x \\ s(n) \in \mathcal{W}_\mathcal{D} \\ g^k \nmid \sigma(n)}} 1 \leq \sum_{\substack{n \leq x \\ g^k \nmid \sigma(n)}} 1 = O\left(x \exp\left(-(\log\log x)^\gamma\right)\right).$$

Next, we work with

$$S_1 = \sum_{\substack{n \in \left(s^{-1}(\mathcal{W}_{\mathcal{D}})\right)(x) \\ \sigma(n) \equiv 0 \bmod g^k}} 1.$$

For

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for some $N \geq 1$, we put

$$B := \sum_{j=0}^{k-1} \varepsilon_j(s(n)) g^j$$

as the number formed by the k-rightmost digits of $s(n)$ so that

$$s(n) \equiv B \bmod g^k.$$

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Let $n \in \left(s^{-1}(\mathcal{W}_{\mathcal{D}})\right)(x)$ with $\sigma(n) \equiv 0 \bmod g^k$. Then, we have

$$n = \sigma(n) - s(n) \equiv -s(n) \equiv -B \bmod g^k.$$

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So, we can relax the condition on S_1 with a congruence condition as follows:

$$S_1 = \sum_{\substack{n \le x \\ s(n) \in \mathcal{W}_{\mathcal{D}} \\ \sigma(n) \equiv 0 \bmod g^k}} 1 \le \sum_{\substack{n \le x \\ n \equiv -B \bmod g^k \\ B \in \mathcal{W}_{\mathcal{D}}(g^k - 1)}} 1 \le \#\mathcal{W}_{\mathcal{D}}(g^k - 1)\left(\left\lfloor \frac{x}{g^k} \right\rfloor + 1\right).$$

We have

$$\#W_{\mathcal{D}}(g^k - 1) = \begin{cases} |\mathcal{D}|^k & \text{if } 0 \in \mathcal{D}, \\ (|\mathcal{D}|^{k+1} - |\mathcal{D}|)/(\mathcal{D}| - 1) & \text{if } 0 \notin \mathcal{D}. \end{cases}$$

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As $k \in \left[\frac{(\log\log x)^\gamma}{\log(g/|\mathcal{D}|)}, 2\frac{(\log\log x)^\gamma}{\log(g/|\mathcal{D}|)}\right]$, we get

$$S_1 \ll x \exp\left(-k \log\left(g/|\mathcal{D}|\right)\right) \ll x \exp\left(-(\log\log x)^\gamma\right).$$

Thus

$$\#(s^{-1}(\mathcal{W}_{\mathcal{D}}))(x) \ll x \exp(-(\log\log x)^\gamma).$$

Revisiting the lemma and the theorem

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where the constant implied by the $\ll$ *notation depends on the choices of* g, A, γ, δ*.*

Theorem (BCDDT, 2023)

Fix $g \geq 3$*,* $\gamma \in (0,1)$*, and a nonempty set* $\mathcal{D} \subsetneq \{0, 1, \ldots, g-1\}$*. For all sufficiently large* x*, the number of* $n \leq x$ *for which* $s(n)$ *has all of its digits in base* g *restricted to digits in* $\mathcal{D}$ *is* $O(x \exp(-(\log\log x)^{\gamma}))$*. That is,*

$$\#\left(s^{-1}(\mathcal{W}_{\mathcal{D}})\right)(x) \ll x \exp(-(\log\log x)^{\gamma}).$$

We also show that $s(n)$ takes infinitely many different values in $\mathcal{W}_{\mathcal{D}}$.

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