Galois cohomology of a real reductive group

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Joint work with Dmitry A. Timashev, Moscow

Thank you for inviting me to give a talk in this workshop.

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For an $\mathbb{R}$-group G, the Galois group Γ acts on $G(\mathbb{C})$, and $G(\mathbb{C})^{\Gamma} = G(\mathbb{R})$.

Abelian Γ-cohomology

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Recall:

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- $\mathrm{B}^1 A = \{{}^\gamma a' - a' \mid a' \in A\} \subseteq \mathrm{Z}^1 A$,
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For an $\mathbb{R}$-torus T, we write

$$\mathrm{H}^1(\mathbb{R}, T) = \mathrm{H}^1\big(\Gamma, T(\mathbb{C})\big).$$

$\mathrm{H}^1(\mathbb{R}, T)$

Notation:

For an $\mathbb{R}$-torus T, we write

- $\mathsf{X}^*(T) = \mathrm{Hom}(T_{\mathbb{C}}, \mathbb{G}_{\mathrm{m},\mathbb{C}})$ (the character group),
- $\mathsf{X}_*(T) = \mathrm{Hom}(\mathbb{G}_{\mathrm{m},\mathbb{C}}, T_{\mathbb{C}})$ (the cocharacter group).

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Proposition (B-Timashev 2021 arXiv)

Let T be an $\mathbb{R}$-torus. The Γ-equivariant homomorphism

$$\mathsf{X}_*(T) \to T(\mathbb{C}), \quad \big(\nu\colon \mathbb{C}^\times \to T(\mathbb{C})\big) \longmapsto \nu(-1)$$

induces a canonical isomorphism

$$\mathrm{H}^1\mathsf{X}_*(T) \xrightarrow{\sim} \mathrm{H}^1(\mathbb{R}, T).$$

Notation: For an $\mathbb{R}$-torus T,

- T_0 is the maximal *compact* (anisotropic) subtorus,
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We have $\mathrm{H}^1(\mathbb{R}, T_1) = \{1\}$ (easy; Theorem 90).

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Write

$$T(\mathbb{R})^{(2)} = \{t \in T(\mathbb{R}) \mid t^2 = 1\}.$$

For $t \in T(\mathbb{R})^{(2)}$ we have $t \cdot {}^\gamma t = t^2 = 1$, whence ${}^\gamma t = t^{-1}$. Thus

$$T(\mathbb{R})^{(2)} \subset \mathrm{Z}^1(\mathbb{R}, T),$$

and we have a canonical homomorphism

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Lemma (B. 1988)

The above homomorphism induces isomorphisms

$$T(\mathbb{R})^{(2)}/T_1(\mathbb{R})^{(2)} \xrightarrow{\sim} \mathrm{H}^1(\mathbb{R}, T);$$
$$T_0(\mathbb{R})^{(2)}/\big(T_0(\mathbb{R})^{(2)} \cap T_1(\mathbb{R})^{(2)}\big) \xrightarrow{\sim} \mathrm{H}^1(\mathbb{R}, T).$$

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If G is an $\mathbb{R}$-group, then $G(\mathbb{C})$ is a Γ-group, and we set

$$\mathrm{H}^1(\mathbb{R}, G) = \mathrm{H}^1\big(\Gamma, G(\mathbb{C})\big).$$

Using Galois cohomology to classify real orbits

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Using H^2 (if necessary), we determine whether $\mathcal{O}$ has real points, and if yes, we find such a point x_0. Set $H = \mathrm{Stab}_G(x_0)$.

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Theorem (Borel-Serre 1964)

There is a canonical bijection

$$\varphi\colon \ker\left[\mathrm{H}^1(\mathbb{R}, H) \to \mathrm{H}^1(\mathbb{R}, G)\right] \longrightarrow \left[\textit{real orbits in } \mathcal{O}\right].$$

Using Galois cohomology to classify real orbits (cont.)

We specify the bijection φ. Write $i\colon H \hookrightarrow G$.
Let $h \in Z^1(\mathbb{R}, H)$ be such that $i_*[h] = [1]$.

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Clearly, for calculations we need *explicit cocycles* representing the cohomology classes.

Relation to arithmetic: H^1 over a number field

Let K be a number field, and G be a connected reductive K-group. The group $\mathrm{H}^1(K,G)$ fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(K,G) & \xrightarrow{\mathrm{ab}^1} & \mathrm{H}^1_{\mathrm{ab}}(K,G) \\ \downarrow{\scriptstyle \mathrm{loc}} & & \downarrow{\scriptstyle \mathrm{loc}} \\ \prod_\infty \mathrm{H}^1(K_v,G) & \xrightarrow{\mathrm{ab}^1} & \prod_\infty \mathrm{H}^1_{\mathrm{ab}}(K_v,G) \end{array}$$

where $\mathrm{H}^1_{\mathrm{ab}}(K,G)$ and $\mathrm{H}^1_{\mathrm{ab}}(K_v,G)$ are certain abelian groups (the *abelian cohomology groups*).

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Moreover, this commutative diagram identifies $\mathrm{H}^1(K,G)$ with the fibered product of $\mathrm{H}^1_{\mathrm{ab}}(K,G)$ and $\prod_\infty \mathrm{H}^1(K_v,G)$ over $\prod_\infty \mathrm{H}^1_{\mathrm{ab}}(K_v,G)$ (B. 1998).

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We see that half of the problem of computing $\mathrm{H}^1(K,G)$ is to compute the H^1 for a reductive $\mathbb{R}$-group.

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Let G be an absolutely simple $\mathbb{R}$-group *of adjoint type*.
Kac 1969: the $\mathbb{R}$-forms of the Lie algebra $\mathrm{Lie}\, G$.
The same as to compute $\mathrm{H}^1(\mathbb{R}, \mathrm{Aut}\, G)$.
We have $G \cong (\mathrm{Aut}\, G)^0$.
The method of Kac gives $\mathrm{H}^1(\mathbb{R}, G)$, and hence the H^1 for all *semisimple $\mathbb{R}$-groups of adjoint type*.

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B-Evenor 2016: $\mathrm{H}^1(\mathbb{R}, G)$, by a method of Borel and Serre.
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Method of Borel and Serre

G a *compact* (hence reductive) connected $\mathbb{R}$-group, that is, $G(\mathbb{R})$ is compact.
$T \subseteq G$ a maximal torus (it is compact).
Then $T(\mathbb{R})^{(2)} \subset \mathrm{Z}^1(\mathbb{R}, T) \subseteq \mathrm{Z}^1(\mathbb{R}, G)$.
The Weyl group $W = W(G_{\mathbb{C}}, T_{\mathbb{C}})$ acts on T and on $T(\mathbb{R})^{(2)}$.

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Theorem (Borel-Serre 1964)

The inclusion map $T(\mathbb{R})^{(2)} \hookrightarrow \mathrm{Z}^1(\mathbb{R}, G)$ induces a canonical bijection

$$T(\mathbb{R})^{(2)}/W \xrightarrow{\sim} \mathrm{H}^1(\mathbb{R}, G).$$

Method of Borel and Serre for noncompact groups

G is a connected reductive $\mathbb{R}$-group, not necessarily compact.
$T_0 \subseteq G$ a maximal *compact* torus.
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Twisted action: $N_0(\mathbb{C}) \curvearrowright T(\mathbb{C})$

$$n * t = n \cdot t \cdot {}^{\gamma}n^{-1} = ntn^{-1} \cdot n\,{}^{\gamma}n^{-1}.$$

Lemma

The above twisted action induces a well-defined action $W_0 \curvearrowright \mathrm{H}^1(\mathbb{R}, T)$.

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In general this action does not preserve $[1] \in \mathrm{H}^1(\mathbb{R}, T)$ and hence does not preserve the group structure in $\mathrm{H}^1(\mathbb{R}, T)$.

Borel-Serre for noncompact groups (cont.)

Theorem (B. 1988)

The inclusion map $T \hookrightarrow G$ induces a bijection

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My co-author Willem de Graaf has implemented this on a computer. For a connected reductive group G (given by its Lie algebra in $\mathfrak{gl}(n, \mathbb{R})$) he can compute a list of representatives $z_1, \ldots, z_m$ of all cohomology classes. Moreover, for a given cocycle $c \in Z^1(\mathbb{R}, G)$, he can determine (using computer) to which of z_i it is cohomologous and find $g \in G(\mathbb{C})$ such that

$$z_i = g \cdot c \cdot \bar{g}^{-1}.$$

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$$z_i = g \cdot c \cdot \bar{g}^{-1}.$$

Furthermore, using nonabelian H^2, he can construct a list $z_1, \ldots, z_m$ also for a not necessarily connected reductive $\mathbb{R}$-group.

Borel-Serre for A_ℓ

When G is a compact simple group of type A_ℓ (that is, isogenous to $\mathrm{SU}_{\ell+1}$), the group $W_0 = W$ has order $(\ell+1)!$. The amount of calculations grows rapidly when ℓ grows!

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By combining the method of Borel and Serre and the method of Kac, we construct a subset

$$\Xi \subset \mathrm{H}^1(\mathbb{R}, T)$$

such that the inclusion map $T \hookrightarrow G$ induces a bijection

$$\Xi/F_0 \xrightarrow{\sim} \mathrm{H}^1(\mathbb{R}, G),$$

where F_0 is a finite group acting on Ξ isomorphic to a subquotient of $Z(G^{\mathrm{sc}})$, and hence of *small order* $\leq \#Z(G^{\mathrm{sc}})$. Here G^{sc} is the universal cover of the commutator subgroup $[G, G]$ of G. For A_ℓ we have $\#F_0 \leq \ell + 1$.

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$R = R(G_{\mathbb{C}}, T_{\mathbb{C}})$ is the root system.
$S = S(G, T, B) = \{\alpha_1, \ldots, \alpha_\ell\}$ is a system of *simple roots* (a basis of R), where $B \subset G_{\mathbb{C}}$ is a Borel subgroup containing $T_{\mathbb{C}}$.
$\alpha_0 \in R$ is the *lowest root* (with respect to S).

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$D = D(R, S)$ is the Dynkin diagram of G (with the set of vertices S).
$\widetilde{D} = \widetilde{D}(R, S)$ is the *extended Dynkin diagram of G* with the set of vertices

$$S \cup \{\alpha_0\} = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}.$$

Linear relation

There is a unique linear relation

$$m_0\alpha_0 + m_1\alpha_1 + \cdots + m_\ell\alpha_\ell = 0$$

normalized such that $m_0 = 1$. All coefficients m_j are positive integers; they are tabulated in Bourbaki-Lie Ch. IV,V,VI, and also in books by Onishchik and Vinberg.

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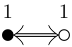
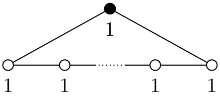
See the extended Dynkin diagrams $\widetilde{D}$ with the coefficients m_j in the tables below. The added vertex α_0 is painted in black.

$\widetilde{D}$ and m_j

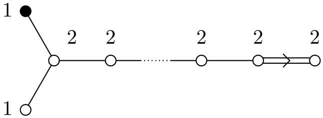
A_1



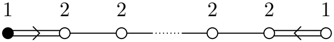
A_ℓ $(\ell \geq 2)$



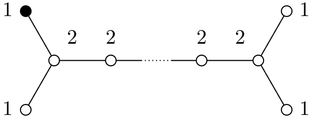
B_ℓ $(\ell \geq 3)$



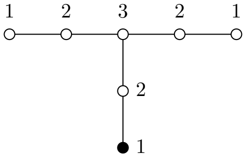
C_ℓ $(\ell \geq 2)$



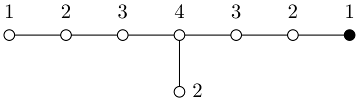
D_ℓ $(\ell \geq 4)$



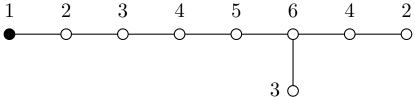
E_6



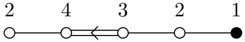
E_7



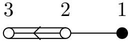
E_8



F_4



G_2



Action of $C = P^\vee/Q^\vee$ on $\widetilde{D}$

Let G be a simple $\mathbb{R}$-group with a compact maximal torus T. Let $G^{\rm sc}$ denote the universal cover of G, and $G^{\rm ad} = G/Z(G)$.
Write $T^{\rm sc} \subset G^{\rm sc}$ for the preimage of T in $G^{\rm sc}$, and $T^{\rm ad} = T/Z(G) \subset G^{\rm ad}$.

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$$X = \mathsf{X}^*(T), \quad P = \mathsf{X}^*(T^{\rm sc}), \quad Q = \mathsf{X}^*(T^{\rm ad});$$
$$X^\vee = \mathsf{X}_*(T), \quad Q^\vee = \mathsf{X}_*(T^{\rm sc}), \quad P^\vee = \mathsf{X}_*(T^{\rm ad}).$$

Then

$$Q \subseteq X \subseteq P, \qquad Q^\vee \subseteq X^\vee \subseteq P^\vee.$$

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The group $C := P^\vee/Q^\vee$ acts on $\widetilde{D}$ effectively.

When $\#C = 2$, the nontrivial element of C acts by the unique nontrivial automorphism of $\widetilde{D}$.

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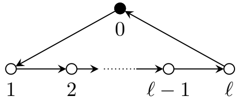
In the cases when $\#C > 2$, see the tables below extracted from Bourbaki-Lie.

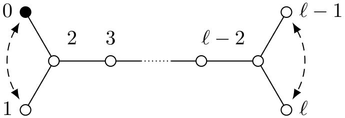
Action of C: A_ℓ and D_{2k}

A_ℓ, $[\omega_1^\vee]$:

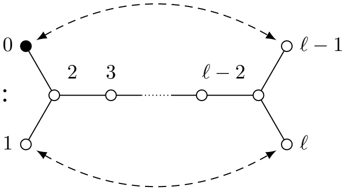


D_ℓ for ℓ even

$[\omega_1^\vee]$:

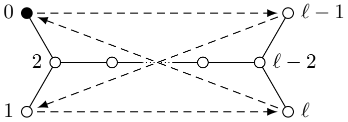


$[\omega_{\ell-1}^\vee]$:

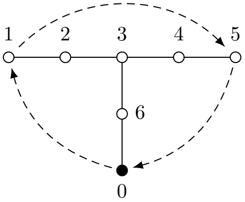


Action of C: D_{2k+1} and E_6

D_ℓ for ℓ odd, $[\omega^\vee_{\ell-1}]$:



E_6, $[\omega^\vee_1]$:



Kac labelings and the theorem of Kac

Definition

A *Kac labeling* of an extended Dynkin diagram $\widetilde{D}$ is a family of numerical labels $q = (q_0, q_1, \ldots, q_\ell)$ with $q_j \in \mathbb{Z}_{\geq 0}$ such that

$$m_0 q_0 + m_1 q_1 + \cdots + m_\ell q_\ell = 2.$$

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Let $\mathcal{K}(\widetilde{D})$ denote the set of Kac labelings of $\widetilde{D}$.

Theorem (Kac 1969)

For a compact *simple* $\mathbb{R}$*-group* $G = G_c$ of adjoint type*, the set of isomorphism classes of inner forms of* G *is in a canonical bijection with the set of orbits* $\mathcal{K}(\widetilde{D})/\mathrm{Aut}(\widetilde{D})$.

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Theorem (version of the theorem of Kac; B-Timashev 2021)

For G as in the theorem of Kac, the set $\mathrm{H}^1(\mathbb{R}, G)$ is in a canonical bijection with the set of orbits $\mathcal{K}(\widetilde{D})/C$.

Theorem of Kac: the bijection

We describe the bijection in the theorem of Kac.
Write $\widetilde{D} = \widetilde{D}(G_{\mathbb{C}}, T_{\mathbb{C}}, B)$. Recall that for $j = 1, \dots, \ell$,

$$\alpha_j \in S \subset R, \quad \alpha_j \colon T_{\mathbb{C}} \to \mathbb{C}^{\times}.$$

The simple roots α_j constitute a basis of $Q = \mathsf{X}^*(T)$.

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Then

$$t_q^2 = 1, \qquad t_q \in T(\mathbb{C})^{(2)} = T(\mathbb{R})^{(2)} \subset \mathrm{Z}^1(\mathbb{R}, G).$$

Theorem of Kac: the bijection (cont.)

Let $G = G_c$ be a *compact* group. We write

$$G_c = (G_{\mathbb{C}}, \sigma_c),$$

where σ_c is the complex conjugation in $G_{\mathbb{C}}$. We set

$$G_q = {}_{t_q}G_c = (G_{\mathbb{C}}, \sigma_q), \quad \text{where} \quad \sigma_q = \mathrm{inn}(t_q) \circ \sigma_c.$$

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To $q \in \mathcal{K}(\widetilde{D})$ we associate $[t_q] \in \mathrm{H}^1(\mathbb{R}, G_c)$. This is the bijection in *our version* of the theorem of Kac.

Non-adjoint simple groups

Let G be an (almost) simple $\mathbb{R}$-group (not necessarily adjoint) having a compact maximal torus T. By a version of the theorem of Kac, we may write $G = G_q \coloneqq {}_{t_q}G_c$, where G_c is a compact group. Write $X = \mathsf{X}^*(T)$. We write $G = G(\widetilde{D}, X, q)$.

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We write $G = G(\widetilde{D}, X, q)$.
Let $\lambda \in X \coloneqq \mathsf{X}^*(T)$. We may write

$$\lambda = \sum_{j=1}^{\ell} c_j \alpha_j \, ,$$

where α_j are the simple roots and where $c_j \in \mathbb{Q}$. For a Kac labeling $p = (p_j)$, we set

$$\langle \lambda, p \rangle = \sum c_j p_j \in \mathbb{Q}.$$

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$$\langle \lambda, p \rangle = \sum c_j p_j \in \mathbb{Q}.$$

If $\lambda \in Q = \mathsf{X}^*(T^{\mathrm{ad}})$, then $c_j \in \mathbb{Z}$ for all $j = 1, \ldots, \ell$, and therefore $\langle \lambda, p \rangle \in \mathbb{Z}$. Thus for $\lambda \in X$, the class

$$\langle \lambda, p \rangle + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$$

depends only on the class of λ in X/Q.

The set $\mathcal{K}(\widetilde{D}, X, q)$

We define a subset $\mathcal{K}(\widetilde{D}, X, q) \subseteq \mathcal{K}(\widetilde{D})$ as follows:

$$(*) \quad \mathcal{K}(\widetilde{D}, X, q) = \left\{ p \in \mathcal{K}(\widetilde{D}) \mid \langle \lambda, p \rangle \equiv \langle \lambda, q \rangle \pmod{\mathbb{Z}} \ \forall [\lambda] \in X/Q \right\}$$

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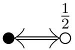
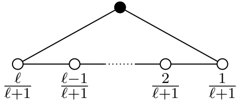
To compute $\langle \lambda, p \rangle$ for a set of generators of X/Q, it suffices to know the coefficients c_j for a set of generators of the finite abelian group $P/Q \supseteq X/Q$. One can find these coefficients in Bourbaki-Lie; see also the tables below.

Coefficients c_j on Dynkin diagrams

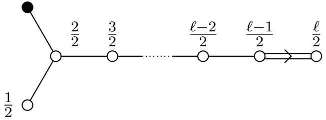
A_1 ω_1:



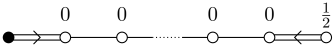
A_ℓ $(\ell \geq 2)$ ω_1:



B_ℓ $(\ell \geq 3)$ ω_ℓ:



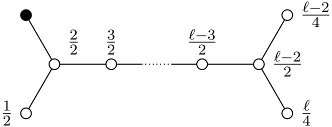
C_ℓ $(\ell \geq 2)$ ω_1:



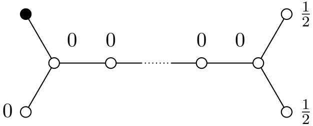
Coefficients c_j on Dynkin diagrams (cont.)

D_ℓ $(\ell \geq 4)$

ω_ℓ :

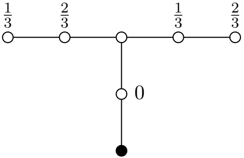


ω_1 :



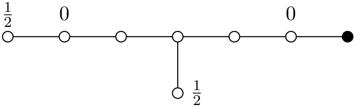
E_6

ω_1:



E_7

ω_7:



$\mathrm{H}^1(\mathbb{R}, G)$ via Kac labelings

The group

$$F = X^\vee/Q^\vee \subseteq P^\vee/Q^\vee = C$$

acts on $\widetilde{D}$ and $\mathcal{K}(\widetilde{D})$ via C.

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Theorem (B-Timashev 2021)

Let $G = G_q$ be an absolutely simple $\mathbb{R}$-group (not necessarily compact or adjoint) having a compact maximal torus T.

(i) *The group F, when acting on $\mathcal{K}(\widetilde{D})$, preserves the subset $\mathcal{K}(\widetilde{D}, X, q)$.*

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(i) *The group F, when acting on $\mathcal{K}(\widetilde{D})$, preserves the subset $\mathcal{K}(\widetilde{D}, X, q)$.*

(ii) *There is a canonical bijection*

$$\mathcal{K}(\widetilde{D}, X, q)/F \xrightarrow{\sim} \mathrm{H}^1(\mathbb{R}, G_q)$$

sending $p = q$ to $[1] \in \mathrm{H}^1(\mathbb{R}, G_q)$.

$\mathrm{H}^1(\mathbb{R}, G)$: the bijection

Write $\widetilde{D} = \widetilde{D}(G_{\mathbb{C}}, T_{\mathbb{C}}, B)$, $\mathfrak{t} = \mathrm{Lie}\, T_{\mathbb{C}}$. Recall that for $j = 1, \ldots, \ell$,

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For

$$G = G_q := {}_{t_q}G_c \quad \text{and} \quad p \in \mathcal{K}(\widetilde{D}, X, q),$$

let $x_q, x_p \in \mathfrak{t}$ be such that

$$d\alpha_j(x_q) = \boldsymbol{i} q_j/2, \quad d\alpha_j(x_p) = \boldsymbol{i} p_j/2 \quad \text{for } j = 1, \dots, \ell.$$

$\mathrm{H}^1(\mathbb{R}, G)$: the bijection (cont.)

Consider the scaled exponential map

$$\mathcal{E}\colon \mathfrak{t} \to T(\mathbb{C}), \quad x \mapsto \exp 2\pi x \quad \text{for } x \in \mathfrak{t}$$

and set

$$t_{p,q} = \mathcal{E}(x_p - x_q) \in T(\mathbb{C}).$$

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One can show that, since $p \in \mathcal{K}(\widetilde{D}, X, q)$, we have $t_{p,q}^2 = 1$, whence

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To $p \in \mathcal{K}(\widetilde{D}, X, q)$ we associate $[t_{p,q}] \in \mathrm{H}^1(\mathbb{R}, G_q)$.

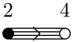
Outer form of a compact group

The case when an *absolutely simple* $\mathbb{R}$-group G
is an *outer* form of a compact group:

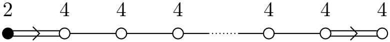
similarly, but one should use
the *twisted affine Dynkin diagrams*,
see below.

Twisted affine Dynkin diagrams and the coefficients m_j

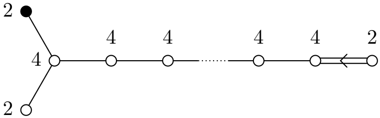
${}^2\mathsf{A}_2$



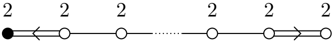
${}^2\mathsf{A}_{2\ell}$ $(\ell \geq 2)$



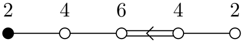
${}^2\mathsf{A}_{2\ell-1}$ $(\ell \geq 3)$



${}^2\mathsf{D}_{\ell+1}$ $(\ell \geq 2)$

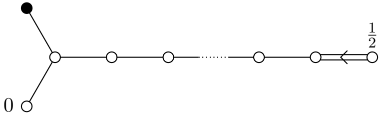


${}^2\mathsf{E}_6$

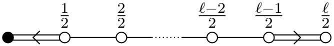


Coefficients c_j on twisted Dynkin diagrams

${}^2\mathsf{A}_{2\ell-1}$ $(\ell \geq 3)$ $\bar{\omega}_1$:



${}^2\mathsf{D}_{\ell+1}$ $(\ell \geq 2)$ $\bar{\omega}_\ell$:

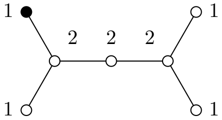


Semisimple groups and reductive groups

The case when G is *semisimple*: see B-Timashev 2021.
The case when G is *reductive*: see B-Timashev 2021 arXiv.

Example

$G = \mathrm{PGO}_{12} := (\mathrm{SO}_{12})^{\mathrm{ad}}$ of type D_6.
The extended Dynkin diagram with the coefficients m_j:



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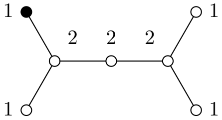


The Kac labelings:

$$\mathcal{K}(\widetilde{D}) = \left\{ q = (q_0, q_1, \ldots, q_6) \;\middle|\; \sum_{j=0}^{6} m_j q_j = 2 \right\}.$$

Inner forms of PGO_{12}

$$
\mathcal{K}(\widetilde{D}) : \quad
\begin{array}{llllll}
{}^{2}_{0}000^{0}_{0} & {}^{0}_{2}000^{0}_{0} & {}^{0}_{0}000^{2}_{0} & {}^{0}_{0}000^{0}_{2} & & \\[1em]
{}^{1}_{1}000^{0}_{0} & {}^{0}_{0}000^{1}_{1} & {}^{1}_{0}000^{1}_{0} & {}^{0}_{1}000^{0}_{1} & {}^{1}_{0}000^{0}_{1} & {}^{0}_{1}000^{1}_{0} \\[1em]
{}^{0}_{0}100^{0}_{0} & {}^{0}_{0}010^{0}_{0} & {}^{0}_{0}001^{0}_{0} & & &
\end{array}
$$

Inner forms of PGO_{12}

$$\mathcal{K}(\widetilde{D}):\quad
\begin{matrix}
{}^{2}_{0}000^{0}_{0} & {}^{0}_{2}000^{0}_{0} & {}^{0}_{0}000^{2}_{0} & {}^{0}_{0}000^{0}_{2} & & \\
{}^{1}_{1}000^{0}_{0} & {}^{0}_{0}000^{1}_{1} & {}^{1}_{0}000^{1}_{0} & {}^{0}_{1}000^{0}_{1} & {}^{1}_{0}000^{0}_{1} & {}^{0}_{1}000^{1}_{0} \\
{}^{0}_{0}100^{0}_{0} & {}^{0}_{0}010^{0}_{0} & {}^{0}_{0}001^{0}_{0} & & &
\end{matrix}$$

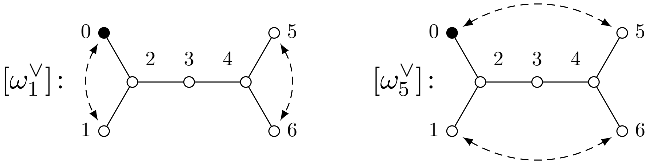
Inner forms of $G = \mathrm{PGO}_{12}$:

$$\mathcal{K}(\widetilde{D})/\mathrm{Aut}(\widetilde{D}):\quad
\begin{matrix}
{}^{2}_{0}000^{0}_{0} & {}^{1}_{1}000^{0}_{0} & {}^{1}_{0}000^{1}_{0} & {}^{0}_{0}100^{0}_{0} & {}^{0}_{0}010^{0}_{0} \\
\mathrm{PGO}_{12} & \mathrm{PGO}_{10,2} & \mathrm{PGO}^{*}_{12} & \mathrm{PGO}_{8,4} & \mathrm{PGO}_{6,6}
\end{matrix}$$

PGO^{*}_{12} is the quaternionic real form of PGO_{12}.

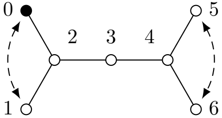
The Galois cohomology of $G = \mathrm{PGO}_{12}$

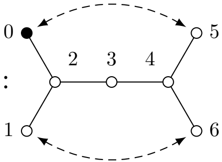
Action of $C \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



The Galois cohomology of $G = \mathrm{PGO}_{12}$

Action of $C \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$[\omega_1^\vee]$: 

$[\omega_5^\vee]$: 

$\mathrm{H}^1(\mathbb{R}, G) \cong \mathcal{K}(\widetilde{D})/C$.

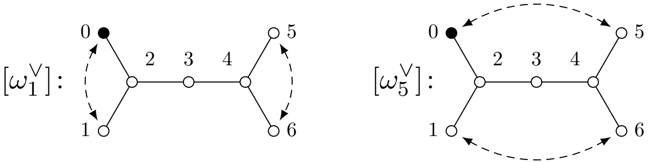
$\mathcal{K}(\widetilde{D})/C$: ${}^{2}_{0}000^{0}_{0}$ ${}^{1}_{1}000^{0}_{0}$ ${}^{1}_{0}000^{1}_{0}$ ${}^{1}_{0}000^{0}_{1}$ ${}^{0}_{0}100^{0}_{0}$ ${}^{0}_{0}010^{0}_{0}$

PGO_{12}^{*} PGO_{12}^{*}

The neutral element is ${}^{2}_{0}000^{0}_{0}$.

The Galois cohomology of $G = \mathrm{PGO}_{12}$

Action of $C \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



$\mathrm{H}^1(\mathbb{R}, G) \cong \mathcal{K}(\widetilde{D})/C$.

$$\mathcal{K}(\widetilde{D})/C : \quad {}^{2}_{0}000^{0}_{0} \quad {}^{1}_{1}000^{0}_{0} \quad {}^{1}_{0}000^{1}_{0} \quad {}^{1}_{0}000^{0}_{1} \quad {}^{0}_{0}100^{0}_{0} \quad {}^{0}_{0}010^{0}_{0}$$

PGO_{12}^{*} PGO_{12}^{*}

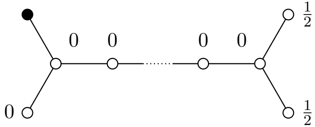
The neutral element is ${}^{2}_{0}000^{0}_{0}$.

Similarly, $\mathrm{H}^1(\mathbb{R}, G_q) \cong \mathcal{K}(\widetilde{D})/C$ for any $q \in \mathcal{K}(\widetilde{D})$, but now the neutral element is the C-orbit of q.

Example: $\mathrm{SO}_{8,4}$

$G = \mathrm{SO}(8,4)$, $\quad q = {}^{0}_{0}100^{0}_{0}$, $X/Q = \{0, [\omega_1]\}$. The coefficients c_j for ω_1 are:

ω_1 :



We have

$$\begin{aligned}
\mathcal{K}(G,X,q) &= \{p \in \mathcal{K}(\widetilde{D}) \mid \langle \omega_1, p\rangle \equiv \langle \omega_1, q\rangle \pmod{\mathbb{Z}}\} \\
&= \{p \in \mathcal{K}(\widetilde{D}) \mid \tfrac{1}{2}p_{\ell-1} + \tfrac{1}{2}p_\ell \equiv \tfrac{1}{2}q_{\ell-1} + \tfrac{1}{2}q_\ell \pmod{\mathbb{Z}}\} \\
&= \{p \in \mathcal{K}(\widetilde{D}) \mid p_{\ell-1} + p_\ell \equiv q_{\ell-1} + q_\ell \pmod{2}\}
\end{aligned}$$

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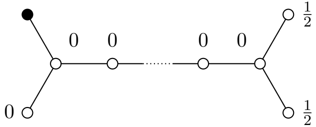

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&= \left\{p \in \mathcal{K}(\widetilde{D}) \mid p_{\ell-1} + p_\ell \equiv q_{\ell-1} + q_\ell \pmod{2}\right\}
\end{aligned}$$

$\mathcal{K}(\widetilde{D}, X, q)$:

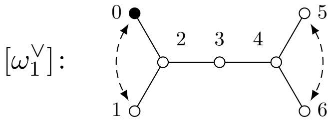
${}^{2}_{0}000^{0}_{0}$ $\quad {}^{0}_{2}000^{0}_{0}$ $\quad {}^{0}_{0}000^{2}_{0}$ $\quad {}^{0}_{0}000^{0}_{2}$

${}^{1}_{1}000^{0}_{0}$ $\quad {}^{0}_{0}000^{1}_{1}$

${}^{0}_{0}100^{0}_{0}$ $\quad {}^{0}_{0}010^{0}_{0}$ $\quad {}^{0}_{0}001^{0}_{0}$

Example: $\mathrm{SO}_{8,4}$ (cont.)

$F = \mathsf{X}^\vee / Q^\vee$ is of order 2 and is generated by $[\omega_1^\vee]$, which acts on $\widetilde{D}$ as follows:



Example: $\mathrm{SO}_{8,4}$ (cont.)

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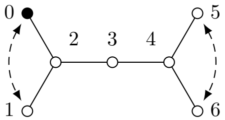
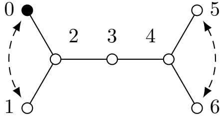


$$\mathrm{H}^1(\mathbb{R}, \mathrm{SO}_{8,4}) \cong \mathcal{K}(\widetilde{D}, X, q)/F: \qquad \begin{matrix} {}^{2}_{0}000^{0}_{0} & {}^{0}_{0}000^{2}_{0} & \\ {}^{1}_{1}000^{0}_{0} & {}^{0}_{0}000^{1}_{1} & \\ {}^{0}_{0}100^{0}_{0} & {}^{0}_{0}010^{0}_{0} & {}^{0}_{0}001^{0}_{0} \end{matrix}$$

Example: $\mathrm{SO}_{8,4}$ (cont.)

$F = \mathsf{X}^\vee / Q^\vee$ is of order 2 and is generated by $[\omega_1^\vee]$, which acts on $\widetilde{D}$ as follows:

$[\omega_1^\vee]$:



$\mathrm{H}^1(\mathbb{R}, \mathrm{SO}_{8,4}) \cong \mathcal{K}(\widetilde{D}, X, q)/F$:

$$\begin{array}{ccc} {}^{2}_{0}000^{0}_{0} & {}^{0}_{0}000^{2}_{0} & \\ {}^{1}_{1}000^{0}_{0} & {}^{0}_{0}000^{1}_{1} & \\ {}^{0}_{0}100^{0}_{0} & {}^{0}_{0}010^{0}_{0} & {}^{0}_{0}001^{0}_{0} \end{array}$$

The neutral element: the class of $q = {}^{0}_{0}100^{0}_{0}$.

$\#\mathrm{H}^1(\mathbb{R}, \mathrm{SO}_{8,4}) = 7$.

Example: SO^*_{12}

$G = \mathrm{SO}^*_{12}, \quad q = {}^{1}_{0}000^{1}_{0}.$

$$\mathcal{K}(\widetilde{D}, X, q): \qquad {}^{1}_{0}000^{1}_{0} \quad {}^{0}_{1}000^{0}_{1} \quad {}^{1}_{0}000^{0}_{1} \quad {}^{0}_{1}000^{1}_{0}$$

Example: SO^*_{12}

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$$\mathrm{H}^1(\mathbb{R}, \mathrm{SO}^*_{12}) \cong \mathcal{K}(\widetilde{D}, X, q)/F: \quad {}^{1}_{0}000^{1}_{0} \quad {}^{1}_{0}000^{0}_{1}$$

The neutral element: the class of $q = {}^{1}_{0}000^{1}_{0}$.
$\#\mathrm{H}^1(\mathbb{R}, \mathrm{SO}^*_{12}) = 2.$

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Thank you!