Algebraic Groups with Good Reduction and Applications

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(joint work with V. Chernousov and A. Rapinchuk)

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1. Groups with good reduction

2. Connections to local-global principles

3. Connections to the genus problem for algebraic groups

4. Some finiteness results

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Then *special fiber* (reduction)

$$\underline{G}^{(v)} = \mathcal{G} \otimes_{\mathcal{O}_v} K^{(v)}$$

is *a connected reductive* group over residue field $K^{(v)}$.

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2. $G = \mathrm{Spin}_n(q)$ has good reduction at v if (over K_v)

$$q \sim \lambda(a_1 x_1^2 + \cdots + a_n x_n^2) \quad \text{with} \quad \lambda \in K_v^{\times}, \ a_i \in \mathcal{O}_v^{\times}$$

(assuming that $\mathrm{char}\, K^{(v)} \neq 2$).

A K-group G' is a *K-form* (or $\overline{K}/K$-form) of G if

$$G' \otimes_K \overline{K} \simeq G \otimes_K \overline{K} \quad \text{(where } \overline{K} \text{ is a sep. closure of } K\text{).}$$

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If A is a central simple algebra of degree n over K, A K-group G' is a *K-form* (or $\overline{K}/K$-form) of G if

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Example.

If A is a central simple algebra of degree n over K, **then**

$$A \otimes_K \overline{K} \simeq M_n(\overline{K})$$

and $G' = \mathrm{SL}_{1,A}$ is a K-form of $G = \mathrm{SL}_n$.

General problem

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To make this question meaningful, one needs to specialize

K, V, and G.

- Previous work has dealt mainly with the case where K is fraction field of *Dedekind ring* R, and V consists of valuations associated with *maximal ideals* of R.

- This situation was first studied in detail by G. Harder (Invent. math. **4**(1967), 165-191) and J.L. Colliot-Thélène & J.J. Sansuc (Math. Ann. **244** (1979), no. 2, 105-134).

- Case $R = \mathbb{Z}$: B.H. Gross (Invent. math. **124**(1996), 263-279) and B. Conrad (Autours des schémas en groupes, Vol. II, 193-253, 2015)

- Case $R = k[x]$: Raghunathan - Ramanathan (Proc. Indian Acad. Sci. Math. Sci. **93** (1984), no. 2-3, 137-145)

- Case $R = k[x, x^{-1}]$: Chernousov - Gille - Pianzola (Amer. J. Math. **134** (2012), no. 6, 1541-1583)

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- R is a finitely generated $\mathbb{Z}$-algebra (or $\mathbb{F}_p$-algebra);
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Then: V corresponds to height one prime ideals of R.

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(If G is absolutely almost simple, $\mathrm{char}\, K = p$ is "good" for G if $p = 0$ or p does not divide order of Weyl group of G. For non-semisimple reductive groups only char. 0 is "good.")

Connections and applications of the Main Conjecture

This conjecture has close connections to:

- Local-global principles for algebraic groups.
- Finiteness properties of unramified cohomology.
- Study of simple algebraic groups having same isomorphism classes of maximal tori (genus problem).
- Analysis of weakly commensurable Zariski-dense subgps and applications to classical problems on locally symmetric spaces (G. Prasad-A. Rapinchuk).

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Thus, study of groups with good reduction occupies a central place in the emerging arithmetic theory of algebraic groups over higher-dimensional fields.

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Set-up: Global-to-local map in Galois cohomology

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One says that the Hasse principle holds if global-to-local map

$$\theta_{G,V} \colon H^1(K,G) \to \prod_{v \in V} H^1(K_v, G)$$

is *injective*.

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Kernel of $\theta_{G,V}$ is called *Tate-Shafarevich set*

$$\text{Ш}(G,V) := \ker \theta_{G,V}.$$

Hasse principle over number fields

Let k = *number field*, V = set of all places of k.

- If G is *simply-connected* or *adjoint* alg. k-group, then

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is injective (i.e. Hasse principle holds).

- For arbitrary alg. k-group G, the map $\theta_{G,V}$ may not be injective, but it is always *proper*; in particular, $\text{Ш}(G,V)$ is finite.

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Our recent results strongly suggest the following properness conjecture for reductive groups over finitely generated fields.

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Properness Conjecture.

If G is a (connected) reductive algebraic K-group, then $\theta_{G,V}$ is proper. In particular, the Tate-Shafarevich set $\text{Ш}(G,V)$ is finite.

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Connection to groups with good reduction:

Proposition 1.

Assume Main Conjecture holds for an absolutely almost simple simply connected K-group G and all divisorial sets of places of K.

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Connection to groups with good reduction:

Proposition 1.

Assume Main Conjecture holds for an absolutely almost simple simply connected K-group G and all divisorial sets of places of K. **Then** *$\theta_{\overline{G},V}$ is proper for corresponding adjoint group $\overline{G}$ and any divisorial set V.*

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• Let G be an absolutely almost simple K-group.

$\mathbf{gen}_K(G) =$ set of isomorphism classes of K-forms G' of G having same K-isomorphism classes of maximal K-tori as G.

Question A. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

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Question B. *When is* $\mathbf{gen}_K(G)$ *finite?*

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(2) *If G is an absolutely almost simple group over a finitely generated field K of "good" characteristic, then* $\mathbf{gen}_K(G)$ *is finite.*

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The theorem remains valid whenever residue field is Hilbertian. (I.R. — work in progress)

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Let K be a finitely generated field, V a divisorial set of places of K, and G an absolutely almost simple simply connected K-group.

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Thus, Main Conjecture provides a uniform approach to both the Properness Conjecture and the finiteness of the genus.

1 Groups with good reduction

2 Connections to local-global principles

3 Connections to the genus problem for algebraic groups

4 Some finiteness results

We have resolved all conjectures for algebraic tori.

- Similar result when $\operatorname{char} K = p > 0$ for tori T for which degree of splitting field $[K_T : K]$ is prime to p.

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Suppose K is a finitely generated field of char. 0, and V is a divisorial set of places. **Then** *for any $d \geqslant 1$, the set of K-isomorphism classes of d-dimensional K-tori having good reduction at all $v \in V$ is finite.*

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Classical proof of this fact for tori over number fields relies on Tate-Nakayama duality, which is not available in general.

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(Proved by P. Gille & L. Moret-Bailly over global fields.)

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(1) *Let D be a central division algebra of exponent 2 over $K = k(x_1, \ldots, x_r)$ where k is a number field or a finite field of characteristic $\neq 2$. Then for $G = \mathrm{SL}_{m,D}$ $(m \geqslant 1)$, we have $|\mathbf{gen}_K(G)| = 1$.*

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(2) *Let $G = \mathrm{SL}_{m,D}$, where D is a central division algebra over a finitely generated field K. Then $\mathbf{gen}_K(G)$ is finite.*

Following Kato, we say K is a 2-dimensional global field if

- $K = k(C)$, with C smooth geometrically integral curve over number field k; or
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- Similar results for some special unitary groups of types A_n, C_n and groups of type G_2.
- **More recently:** similar result for $\widetilde{\mathrm{SU}}_n(D,h)$, with D a quaternion division algebra over $K = k(C)$ and h a skew-hermitian form over D (I.R. — work in progress)

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- Some further finiteness results over function fields of rational surfaces and certain Severi-Brauer varieties over number fields.

Some results on Properness Conjecture

- $\mathrm{PSL}_{1,A}$ over arbitrary finitely generated fields.
- K a 2-dimensional global field and
  - $G = \mathrm{SO}_n(q)$ $(n \geqslant 5)$;
  - G of type G_2;
  - $G = \mathrm{SU}_n(L/K, h)$, L/K quadratic extension, h nondegenerate hermitian form of $\dim \geqslant 2$;
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- K a purely transcendental extension or function field of Severi-Brauer variety over number field and G of type G_2.

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