Chromatic Symmetric Functions and Sign-Reversing Involutions

Bruce Sagan
Michigan State University
www.math.msu.edu/~sagan

joint work with Zachary Hamaker and Vincent Vatter

BIRS Workshop on Interactions between Hessenberg Varieties, Chromatic Functions, and LLT Polynomials

Sign-reversing involutions

The $(3 + 1)$-free Conjecture

The coefficient of e_n

Other results and future work

Let S be a finite set. An *involution* on S is a bijection $\iota : S \to S$ with

$$\iota^2 = \mathrm{id}\,.$$

So, viewed as a permutation of S, all cycles of ι are of length 1 or 2. Suppose S is *signed* so there is a function

$$\mathrm{sgn} : S \to \{+1, -1\}.$$

Call ι a *sign-reversing involution* if

1. for all 1-cycles (s) we have $\mathrm{sgn}\, s = +1$, and
2. for all 2-cycles (s, t) we have $\mathrm{sgn}\, s = -\mathrm{sgn}\, t$.

If ι is a sign-reversing involution on S then

$$\sum_{s \in S} \mathrm{sgn}\, s = \# S^{\iota}$$

where $\#$ is cardinality and S^{ι} is the fixed-point set of ι. Suppose R is a ring and weight S by a function $\mathrm{wt} : S \to R$. If ι is weight preserving in that $\mathrm{wt}\, \iota(s) = \mathrm{wt}\, s$ for all $s \in S$, then

$$\sum_{s \in S} (\mathrm{sgn}\, s)(\mathrm{wt}\, s) = \sum_{s \in S^{\iota}} \mathrm{wt}\, s.$$

Let

$$[n] = \{1, 2, \ldots, n\}.$$

And denote *symmetric difference* of sets A, B by

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Proposition

If $n \geq 1$ then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof.

Let $S = \{A \subseteq [n]\}$. Give S the sign function

$$\operatorname{sgn} A = (-1)^{\# A}.$$

$$\therefore \sum_{A \in S} \operatorname{sgn} A = \sum_{k=0}^{n} \sum_{A \in S, \# A = k} (-1)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}.$$

Define involution $\iota : S \to S$ by $\iota(A) = A \Delta \{n\}$. So ι has no fixed points and is sign reversing. Thus the sum equals $\# S^{\iota} = 0$. $\square$

Let $G = (V, E)$ be a graph. Given a set S, a vertex coloring $\kappa : V \to S$ is *proper* if

$$uv \in E \implies \kappa(u) \neq \kappa(v).$$

Let $\mathbb{P}$ be the positive integers and $\mathbf{x} = \{x_1, x_2, \ldots\}$. Given a proper vertex coloring $\kappa : V \to \mathbb{P}$ we let

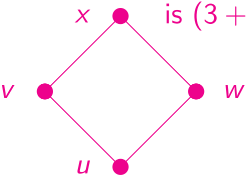
$$\mathbf{x}^{\kappa} = \prod_{v \in V} x_{\kappa(v)}.$$

Stanley's *chromatic symmetric function* is

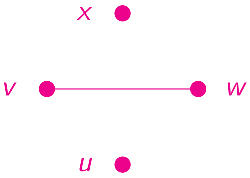
$$X(G) = X(G; \mathbf{x}) = \sum_{\kappa} \mathbf{x}^{\kappa}$$

where the sum is over all proper $\kappa : V \to \mathbb{P}$.

Let $(P, \leq_P)$ be a poset. Say P is *$(m+n)$-free* if it contains no induced subposet isomorphic to $[m] \uplus [n]$. The *incomparability graph of P* is $\mathrm{inc}(P) = (P, E)$ where $uv \in E$ if neither $u \leq_P v$ nor $v \leq_P u$. Let $\{e_\lambda\}$ and $\{s_\lambda\}$ be the elementary and Schur bases for symmetric functions, respectively. Given a basis $\{b_\lambda\}$, a symmetric function $f(\mathbf{x})$ is *b-positive* if the coefficients in its expansion in this basis are nonnegative.

Ex. $P =$ 

is $(3+1)$-free

$\mathrm{inc}(P) =$ 

Conjecture (Stanley-Stembridge $(3+1)$-free Conjecture)

If P is a $(3+1)$-free poset then $X(\mathrm{inc}(P); \mathbf{x})$ is e-positive.

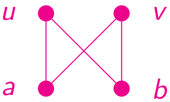
The Method.

1. Expand $X(\mathrm{inc}(P))$ in terms of s_λ using Gasharov's P-tableaux.
2. Expand the s_λ in terms of e_μ using Jacobi-Trudi determinants.
3. Use a sign-reversing involution to cancel the negative terms.

Given poset $(P, \leq_P)$, a *P-tableau T of shape λ* is a bijective filling of the Young diagram of λ with the elements of P such that

1. rows are increasing with respect to $\leq_P$, and
2. adjacent elements in a column are nondecreasing with respect to $\leq_P$.

Ex.

$P = P_{2,2} =$ 

Some P-tableaux:

a	u
b	v

b	v
a	
u	

Some non-P-tableaux:

a	b
u	v

b	v
u	
a	

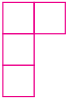
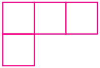
Let $\mathrm{PT}(P)$ and $\mathrm{PT}_\lambda(P)$ be the set of all P-tableau and those of shape λ, respectively.

Theorem (Gasharov)

If P is $(3+1)$-free and $X(\mathrm{inc}(P)) = \sum_\lambda c_\lambda s_\lambda$ then

$$c_\lambda = \#\,\mathrm{PT}_\lambda(P).$$

The *transpose* of partition λ is λ^t = diagonally reflect λ.

Ex. If $\lambda =$  then $\lambda^t =$ .

Theorem (dual Jacobi-Trudi determinant)

If $\lambda = (\lambda_1, \lambda_2, \ldots)$ then $s_{\lambda^t} = \begin{vmatrix} e_{\lambda_1} & e_{\lambda_1+1} & \cdots \\ e_{\lambda_2-1} & e_{\lambda_2} & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix}$.

So writing $X(\mathrm{inc}(P))$ first in s_λ and then in e_μ has signed coefficients which count pairs (T, π) where $T \in \mathrm{PT}_\lambda(P)$ and $\pi \in \mathfrak{S}_{\lambda_1}$ is the permutation from the determinant expansion.

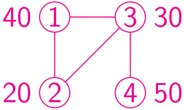
Ex. If $P = P_{2,2}$ then $\#\,\mathrm{PT}_\lambda(P) = 4$ for $\lambda = (2^2), (2, 1^2), (1^4)$.

$$\begin{aligned} X(\mathrm{inc}(P)) &= 4s_{2^2} + 4s_{2,1^2} + 4s_{1^4} \\ &= 4\begin{vmatrix} e_2 & e_3 \\ e_1 & e_2 \end{vmatrix} + 4\begin{vmatrix} e_3 & e_4 \\ e_0 & e_1 \end{vmatrix} + 4e_4 \\ &= 4e_{2^2} - 4e_{3,1} + 4e_{3,1} - 4e_4 + 4e_4 \\ &= 4e_{2^2}. \end{aligned}$$

Let G be a graph with $V = [n]$ and $\kappa : [n] \to \mathbb{P}$ be a proper coloring. An *ascent* of κ is an edge ij with

1. $i < j$, and
2. $\kappa(i) < \kappa(j)$.

Let $\operatorname{asc}\kappa$ be the number of ascents of κ.

Ex. 

ascents: 23 since $\kappa(2) = 20 < 30 = \kappa(3)$,
34 since $\kappa(3) = 30 < 50 = \kappa(4)$.
So $\operatorname{asc}\kappa = 2$.

If t is a variable then the Shareshian-Wachs *chromatic quasisymmetric function* of a graph G with $V = [n]$ is

$$X(G; \mathbf{x}, t) = \sum_{\kappa: V \to \mathbb{P} \text{ proper}} t^{\operatorname{asc}\kappa} \mathbf{x}^{\kappa}.$$

Theorem (Shareshian-Wachs)

If P is a natural unit interval order (NUIO) then $X(\operatorname{inc}(P); \mathbf{x}, t)$ is symmetric.

Conjecture (Shareshian-Wachs)

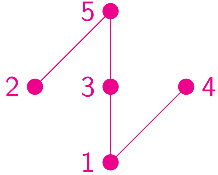
If P is a NUIO then $X(\operatorname{inc}(P); \mathbf{x}, t)$ is e-positive.

Let P be an NUIO, and so a poset on $[n]$, and let T be a P-tableau. An *inversion* in T is a pair $i, j \in [n]$ with

1. $i < j$,
2. i is in a lower row than j, and
3. i and j are incomparable in P.

Let $\mathrm{Inv}\ T$ be the set of inversions of T and $\mathrm{inv}\ T = \#\,\mathrm{Inv}\ T$.

Ex.

$P =$ 

$T =$

1	3	5
2		
4		

$\mathrm{Inv}\ T = \{23, 45\}$

Theorem (Shareshian-Wachs)

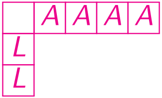
If P is an NUIO and $X(\mathrm{inc}(P); \mathbf{x}, t) = \sum_\lambda c_\lambda(t) s_\lambda$ then

$$c_\lambda(t) = \sum_{T \in \mathrm{PT}_\lambda(P)} t^{\mathrm{inv}\ T}.$$

Let $\#P = n$ and $\lambda \vdash n$. The e_h of largest subscript appearing in the determinant for s_λ is at the end of the first row. And in that case h is the hooklength of the $(1,1)$ box of the diagram of λ. So if $h = n$ then λ is a hook. Furthermose e_n only occurs as the term in the determinant corresponding to the permutation given by $\pi = c, 1, 2, \ldots, c-1$ where $c = \lambda_1$. So if λ is a hook then let the *sign* of a P-tableau T of shape λ be

$$\operatorname{sgn} T = \operatorname{sgn} \lambda = (-1)^{c-1}.$$

If λ is a hook then its *arm* and *leg* are the boxes in the first row, respectively first column, except (1,1).

Ex. $\lambda =$ 

$$s_\lambda = \begin{vmatrix} e_3 & e_4 & e_5 & e_6 & e_7 \\ e_0 & e_1 & e_2 & e_3 & e_4 \\ 0 & e_0 & e_1 & e_2 & e_3 \\ 0 & 0 & e_0 & e_1 & e_2 \\ 0 & 0 & 0 & e_0 & e_1 \end{vmatrix}$$

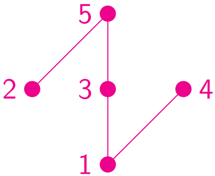
$\pi = 51234$ $\quad$ $\operatorname{sgn} \lambda = (-1)^{5-1} = 1.$

$A =$ arm, $L =$ leg.

Let P be an NUIO on $[n]$ and T be a P-tableau of hook shape. Call $k \in [n]$ *movable* in T if it can be moved from the arm to the leg of T or vice-versa so that

1. the resulting tableau T' is a P-tableau, and
2. $\operatorname{Inv} T = \operatorname{Inv} T'$.

Ex.

$P =$ 

$T = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$ $\qquad \operatorname{Inv} T = \{23, 45\}$

3 is movable with $T' = \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$. 5 is movable with $T' = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$.

2 and 4 are not movable.

Lemma (Hamaker-S-Vatter)

If k is movable in T, then there is a unique position to which it can be moved.

If k is movable in T then let T^k be the result of moving k. Define a map ι on P-tableau T of hook shape by

$$\iota(T) = \begin{cases} T^k & \text{if } k \text{ is the smallest integer which is movable in } T, \\ T & \text{if no element in } T \text{ is movable.} \end{cases}$$

Theorem (Hamaker-S-Vatter)

Let P be any NUIO on $[n]$.

1. *ι is a sign-reversing, Inv-preserving, involution on hook P-tableaux.*
2. *If T is fixed by ι then it has shape (1^n).*
3. *The coefficient $c_n(t)$ of e_n in $X(\mathrm{inc}(P); \mathbf{x}, t)$ has nonnegative coefficients. It is the generating function by inv of P-tableau of column shape with no moveable elements.*

Acyclic orientations.

An *orientation* O of a graph G is obtained by replacing each edge $uv \in G$ by one of the arcs $\vec{uv}$ or $\vec{vu}$. Call O *acyclic* if it has no directed cycles. If $V = [n]$ then an *ascent* of O is an arc $\vec{uv}$ with $u < v$, and we let $\operatorname{asc} O$ be the number of ascents of O.

Theorem (Stanley,Shareshian-Wachs)

If P is an NUIO on $[n]$ and $X(\mathrm{inc}(P); \mathbf{x}, t) = \sum_\lambda c_\lambda(t) e_\lambda$, then

$$\sum_{\lambda \text{ with } m \text{ parts}} c_\lambda(t) = \sum_{O \text{ with } m \text{ sinks}} t^{\operatorname{asc} O}.$$

So if $\lambda = (n)$ then $c_n(t) = \sum_{O \text{ with } 1 \text{ sink}} t^{\operatorname{asc} O}$. Given a P-tableau T of column shape we define an orientation O of $G = \mathrm{inc}\, P$ by orienting each edge uv of G so that

$$\vec{uv} \text{ is an arc of } O \text{ iff } uv \in \mathrm{Inv}\, T.$$

Theorem (Hamaker-S-Vatter)

For any NUIO and $m \geq 0$, the map $T \mapsto O$ is an inv-asc preserving bijection from P-tableaux of column shape with m movable elements to acyclic orientations of $\mathrm{inc}(P)$ with $m + 1$ sinks.

Related work.

Shareshian and Wachs used an involution which is similar but not equivalent to the involution for e_n in their determination of the coefficient of p_n (power sum symmetric function) in $X(\mathrm{inc}(P); \mathbf{x}, t)$.

There have been other applications of The Method. The *height* of a poset P, $\mathrm{ht}\, P$, is the number of elements in a longest chain. If P is an NUIO then $\mathrm{ht}\, P$ is the bounce number of the corresponding Dyck path. Harada and Precup proved the $(3+1)$-free conjecture for $X(\mathrm{inc}(P); \mathbf{x}, t)$ when $\mathrm{ht}\, P = 2$ using Hessenberg varieties. Cho and Huh gave a combinatorial proof of this result using The Method. Cho and Hong used The Method to prove the $(3+1)$-free conjecture for $X(\mathrm{inc}(P); \mathbf{x})$ when $\mathrm{ht}\, P = 3$. Finding a proof for $X(\mathrm{inc}(P); \mathbf{x}, t)$ when $\mathrm{ht}\, P = 3$ is still open but certain special cases were done using involutions by Cho and Hong, and by Wang using the inverse Kostka matrx in place of the Jacobi-Trudi determinant.

References

1. Cho, S.; Hong, J. Positivity of chromatic symmetric functions associated with Hessenberg functions of bounce number 3, *Electron. J. Combin.* 29 (2022), Paper No. 2.19, 37 pp.

2. Cho, S.; Huh, J.; On e-positivity and e-unimodality of chromatic quasi-symmetric functions, *SIAM J. Discrete Math.* 33 (2019), 2286—2315.

3. Harada, M.; Precup, M. The cohomology of abelian Hessenberg varieties and the Stanley-Stembridge conjecture, *Algebr. Comb.* 2 (2019), 1059—1108.

4. Sagan, B.; Vatter, V.; Bijective proofs of proper coloring theorems, *Amer. Math. Monthly* 128 (2021), 483—499.

5. Shareshian, J.; Wachs, M. Chromatic quasisymmetric functions, *Adv. Math.*, 295 (2016), 497–551.

6. Stanley, R. A symmetric function generalization of the chromatic polynomial of a graph, *Adv. Math.*, 111 (1995), 166–194.

7. Stanley, R.; Stembridge, J. On immanants of Jacobi-Trudi matrices and permutations with restricted position, *J. Combin. Theory Ser. A* 62 (1993), no. 2, 261–279.

8. Wang, S.; The e-positivity of the chromatic symmetruc functions and the inverse Kostka matrix, arXiv 2210.07567.

THANKS FOR LISTENING!