Parabolic Lusztig varieties and chromatic symmetric functions

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Outline

Lusztig varieties

Parabolic Lusztig varieties

Forgetting to $\mathbb{P}^{n-1}$

Schubert varieties and Lusztig Varieties

$G = GL_n(\mathbb{C})$, B Borel subgroup, G/B is the flag variety, whose elements are denoted either gB or $V_\bullet = V_1 \subset \ldots V_{n-1} \subset \mathbb{C}^n$. We let $w \in S_n$ and write

$$r_{ij}(w) := |\{k; k \leq i, w(k) \leq j\}|$$

Schubert varieties

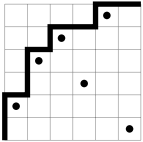
$$\Omega_w(F_\bullet) := \{V_\bullet; \dim V_i \cap F_j \geq r_{ij}(w)\}$$
$$\Omega_w(g_0 B) := \{gB; g \in g_0 \overline{B\dot{w}B}\}.$$

Lusztig varieties

$$\mathcal{Y}_w(X) := \{V_\bullet; \dim XV_i \cap V_j \geq r_{ij}(w)\},$$
$$\mathcal{Y}_w(X) := \{gB; g^{-1}Xg \in \overline{B\dot{w}B}\},$$

Codominant permutations and Hessenberg varieties

Codominant permutations



A permutation w is codominant if w is 312-avoiding. Analogously, there exists a Hessenberg function $\mathbf{m}\colon [n] \to [n]$ such that w is the greatest (in the Bruhat sense) permutation satisfying $w(i) \leq \mathbf{m}(i)$ for every $i \in [n]$. We write $w_{\mathbf{m}}$ for this permutation.

Hessenberg varieties

For $w_{\mathbf{m}}$ we have

$$\mathcal{Y}_{w_{\mathbf{m}}}(X) := \{(X, V_\bullet); XV_i \subseteq V_{\mathbf{m}(i)}\}.$$

Lusztig varieties and characters of Kazdhan-Lusztig elements

When X is a regular semisimple matrix, we have that $IH^*(\mathcal{Y}_w(X))$ has a natural structure of S_n-module, and its Frobenius character is related to the characters of the Kazdhan-Lusztig basis elements C'_w of the Hecke Algebra.

Theorem (Lusztig, A-N)

We have that

$$\mathrm{ch}(IH^*(\mathcal{Y}_w(X)) = \sum_\lambda \chi^\lambda(q^{\frac{\ell(w)}{2}} C'_w) s_\lambda =: \mathrm{ch}(q^{\frac{\ell(w)}{2}} C'_w).$$

Theorem (Brosnan-Chow, Guay-Paquet)

We have that

$$\mathrm{ch}(IH^*(\mathcal{Y}_{w_{\mathbf{m}}}(X)) = \omega(\mathrm{csf}_q(G_{\mathbf{m}})).$$

Iwahori-Hecke algebras

- The Iwahori-Hecke algebra H_n is the algebra over $\mathbb{C}(q^{\frac{1}{2}})$ generated by T_{s_i} with the following relations

$$\begin{aligned} T_{s_i}^2 =& (q-1)T_{s_i} + q \\ T_{s_i}T_{s_{i+1}}T_{s_i} =& T_{s_{i+1}}T_{s_i}T_{s_{i+1}} \\ T_{s_i}T_{s_j} =& T_{s_j}T_{s_i} && \text{if } |i-j|>1. \end{aligned}$$

  It is a q-deformation of the group algebra $\mathbb{C}[S_n]$.
- If $w = s_{i_1}s_{i_2}\ldots s_{i_m}$ is a reduced expression for w, then we define $T_w = T_{s_{i_1}}T_{s_{i_2}}\cdots T_{s_{i_m}}$.
- There is an involution $\iota\colon H_n \to H_n$, such that $\iota(T_w) = T_{w^{-1}}^{-1}$ and $\iota(q^{\frac{1}{2}}) = q^{-\frac{1}{2}}$
- The irreducible representations $H_n \to End(V_\lambda)$ are analogues of the irreducible representations of S_n. The difference is that V_λ is a $\mathbb{C}(q^{\frac{1}{2}})$-vector space.

Kazdhan-Lusztig basis

- There exists a distinguished basis C'_w of H_n. Defined by the following properties.

$$\iota(C'_w) = C'_w$$
$$q^{\frac{\ell(w)}{2}} C'_w = \sum_{z \leq w} P_{z,w}(q) T_z$$

  where $P_{z,w}(q)$ is a polynomial in q. The polynomial $P_{z,w}$ is called the Kazdhan-Lusztig polynomial.
- These elements C'_w and the polynomials $P_{z,w}$ are closely related to the Schubert variety Ω_w.
- In a way, the polynomial $P_{z,w}$ measures how much Ω_w is singular along Ω°_z.

Combinatorial interpretations of $\mathrm{ch}(q^{\frac{\ell(w)}{2}} C'_w)$

Theorem (Clearman-Hyatt-Shelton-Skandera)

When w is a smooth permutation (Ω_w is smooth, w is 3412 and 4231 avoiding), there are combinatorial interpretations of the coefficients of $\mathrm{ch}(q^{\frac{\ell(w)}{2}} C'_w)$ for some basis of the symmetric algebra. In particular,

$$\mathrm{ch}(q^{\frac{\ell(w_{\mathbf{m}})}{2}} C'_{w_{\mathbf{m}}}) = \omega(\mathrm{csf}_q(G_{\mathbf{m}})).$$

Moreover, for every smooth permutation w, there exists a Hessenberg function $\mathbf{m}$ such that

$$\mathrm{ch}(q^{\frac{\ell(w)}{2}} C'_w) = \omega(\mathrm{csf}_q(G_{\mathbf{m}})).$$

Conjecture (Haiman)

The character $\mathrm{ch}(q^{\frac{\ell(w)}{2}} C'_w)$ is h-positive for every $w \in S_n$.

Outline

Lusztig varieties

Parabolic Lusztig varieties

Forgetting to $\mathbb{P}^{n-1}$

Parabolic

- The Lusztig variety

$$\mathcal{Y}_w(X) = \{V_\bullet; XV_i \cap V_j \geq r_{ij}(w)\}$$

is defined in analogy to the Schubert variety ($F_\bullet$ is a fixed flag)

$$\Omega_w(F_\bullet) = \{V_\bullet; V_i \cap F_j \geq r_{ij}(w)\}.$$

- The parabolic Schubert varieties are also defined in terms of a fixed flag $F_\bullet$. Let $J \subset \{1, \ldots, n-1\}$, with $J^c = \{i_1, \ldots, i_k\}$, be a subset of the set of simple transpositions of S_n and w a permutation in ${}^J S_n$, we define

$$\Omega_{w,J}(F_\bullet) = \{V_\bullet; \dim V_i \cap F_j \geq r_{i,j}(w), i \notin J\}.$$

The flag $V_\bullet$ above is a partial flag

$$0 = V_0 \subset V_{i_1} \subset \cdots \subset V_{i_k} \subset \mathbb{C}^n.$$

Difficulty constructing parabolic Lusztig varieties

- If we try to extend this to construct parabolic Lusztig varieties and simply compare $XV_\bullet$ and $V_\bullet$ we will get very few varieties.

- Consider $\mathrm{Gr}(1, n) = \mathbb{P}^{n-1}$. The only possible relative position of XV_1 and V_1 are
  1. $\{V_1; \dim XV_1 \cap V_1 \geq 1\}$, which is the set of the points $(0 : \ldots : 0 : 1 : 0 : \ldots : 0)$.
  2. $\{V_1; \dim XV_1 \cap V_1 \geq 0\}$, which is all $\mathbb{P}^{n-1}$.

Extending the flag

Let $(X, V_\bullet)$ be a pair in $G \times G/P_J$, where

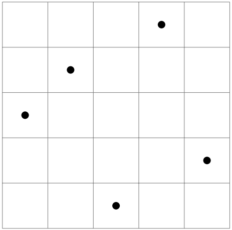
$$V_\bullet = (V_0 \subset V_{i_1} \subset V_{i_2} \subset \ldots \subset V_{i_k} \subset \mathbb{C}^n).$$

We construct the flag $V'_\bullet$ by coalescing (we remove vector spaces with the same dimension) the flag

$$0 \subseteq V_{i_1} \cap XV_{i_1} \subseteq V_{i_1} \cap XV_{i_2} \subseteq \ldots V_{i_1} \cap XV_{i_k} \subseteq V_{i_1} \subseteq$$
$$\subseteq V_{i_1} + (V_{i_2} \cap XV_{i_1}) \subseteq V_{i_1} + (V_{i_2} \cap XV_{i_2}) \subseteq \ldots V_{i_1} + V_{i_2} \cap XV_{i_k} \subseteq V_{i_2} \subseteq$$
$$\vdots$$
$$\subseteq V_{i_j} + (V_{i_{j+1}} \cap XV_{i_1}) \subseteq V_{i_j} + (V_{i_{j+1}} \cap XV_{i_2}) \subseteq \ldots \subseteq V_{i_j} + (V_{i_{j+1}} \cap XV_{i_k}) \subseteq V_{i_{j+1}}$$
$$\vdots$$
$$\subseteq V_{i_k} + XV_{i_1} \subseteq V_{i_k} + XV_{i_2} \subseteq \ldots \subseteq V_{i_k} + XV_{i_k} \subseteq V_n = \mathbb{C}^n.$$

We iterate this construction, until it stabilizes and we get a pair $(X, \widetilde{V_\bullet}) \in G \times G/P_{\widetilde{J}}$.

Example in $\mathrm{Gr}(2,5)$, $J = \{1,3,4\}$



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- $V_4 = V_2 + XV_2.$
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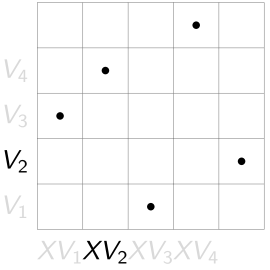

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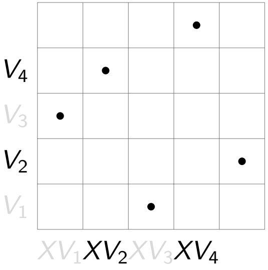

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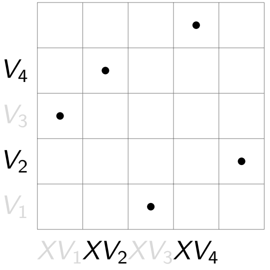

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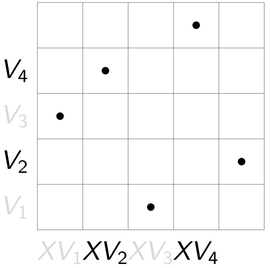

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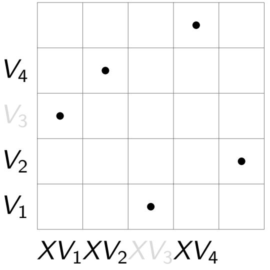

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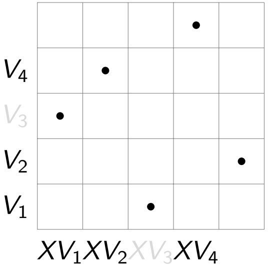

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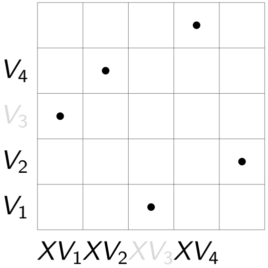

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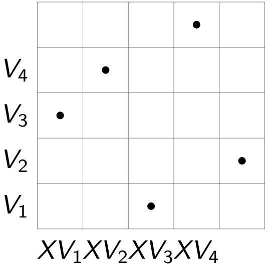

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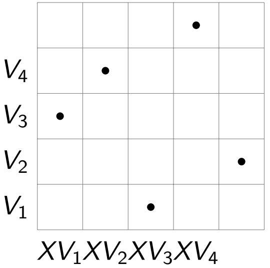

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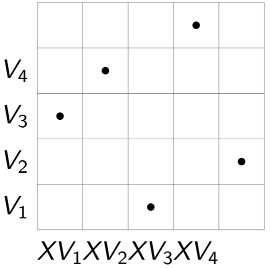

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Example in $\mathrm{Gr}(2,4)$

$$\mathcal{Y}_{1234,J}(X) = \{V_2; XV_2 = V_2\},$$
$$\Omega_{1234,J,F_\bullet} = \{V_2; V_2 = F_2\},$$
$$\mathcal{Y}_{1324,J}(X) = \left\{ V_2; \begin{array}{l} \dim(V_2 \cap XV_2 \cap X^2V_2) \geq 1 \\ \dim(V_2 + XV_2 + X^2V_2) \leq 3 \end{array} \right\},$$
$$\Omega_{1324,J,F_\bullet} = \{V_2; F_1 \subset V_2 \subset F_3\},$$
$$\mathcal{Y}_{1342,J}(X) = \{V_2; \dim(V_2 \cap XV_2 \cap X^2V_2) \geq 1\},$$
$$\Omega_{1342,J,F_\bullet} = \{V_2; F_1 \subset V_2\},$$
$$\mathcal{Y}_{3124,J}(X) = \{V_2; \dim(V_2 + XV_2 + X^2V_2) \leq 3\},$$
$$\Omega_{3124,J,F_\bullet} = \{V_2; V_2 \subset F_3\},$$
$$\mathcal{Y}_{3142,J}(X) = \{V_2; \dim(XV_2 \cap V_2) \geq 1\},$$
$$\Omega_{3142,J,F_\bullet} = \{V_2; \dim(V_2 \cap F_2) \geq 1\},$$
$$\mathcal{Y}_{3412,J}(X) = \mathrm{Gr}(2,4) = \Omega_{3412,J,F_\bullet},$$

Frobenius characters

$$\begin{aligned}
\operatorname{ch}(IH^*(\mathcal{Y}_{1234}(X))) &= h_{2,2},\\
\operatorname{ch}(IH^*(\mathcal{Y}_{1324}(X))) &= (1+q)h_{2,1,1},\\
\operatorname{ch}(IH^*(\mathcal{Y}_{1342}(X))) &= (1+q+q^2)h_{3,1},\\
\operatorname{ch}(IH^*(\mathcal{Y}_{3124}(X))) &= (1+q+q^2)h_{3,1},\\
\operatorname{ch}(IH^*(\mathcal{Y}_{3142}(X))) &= (1+q+q^2+q^3)h_4 + (q+q^2)h_{3,1},\\
\operatorname{ch}(IH^*(\mathcal{Y}_{3412}(X))) &= (1+q+2q^2+q^3+q^4)h_4 = \binom{4}{2}_q h_4.
\end{aligned}$$

Example in $\mathrm{Gr}(1, n) = \mathbb{P}^{n-1}$

- $J = \{2, \ldots, n-1\}$, $G/P_J = \mathbb{P}^{n-1}$
- $$^J S_n = \{12\ldots n, 213\ldots n, 231\ldots n, \ldots, 234\ldots n1\}.$$
  Define $w_k := 23\ldots k1(k+1)\ldots n$ $(w_1 = 12\ldots n)$
- $$\mathcal{Y}_{w_k,J}(X) = \{V_1; \dim(V_1 + XV_1 + \ldots + X^k V_1) \leq k\},$$
  or, equivalently,
  $$\mathcal{Y}_{w_k,J}(X) = \left\{ V_1; \begin{array}{l} V_1 \text{ is contained is a } k\text{-dimensional} \\ \text{subspace invariant by } X \end{array} \right\}.$$
- If X is a diagonal matrix, then $\mathcal{Y}_{w_k,J}(X)$ is the union of coordinates $(k-1)$-planes of $\mathbb{P}^{n-1}$.

Example in $\mathrm{Gr}(1, n) = \mathbb{P}^{n-1}$

- Changing notation and defining $\mathcal{H}_k = \mathcal{Y}_{w_{n-k}, J}(X)$, which is the union of codimension k coordinates planes. (k=0,. . . , n-1)
- $\mathcal{H}_k$ is the union of the closures of the orbits of codimension k via the action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^{n-1}$.
- The normalization $\widetilde{\mathcal{H}}_k$ of $\mathcal{H}_k$ is the union of $\binom{n}{k}$ copies of $\mathbb{P}^{n-k-1}$.
- Hence the intersection cohomology of $\mathcal{H}_k$ is the cohomology $H^*(\widetilde{\mathcal{H}}_k)$. The latter has a natural structure of S_n module and

$$\mathrm{ch}(H^*(\widetilde{\mathcal{H}}_k)) = [n-k]_q h_{n-k,k}.$$

Decomposition theorem

Theorem

Consider $f : \mathcal{B} \to \mathcal{B}_J$ the forgetful map and let X be a regular semisimple invertible matrix. Then, we have a splitting of S_n-modules

$$IH^*(\mathcal{Y}_w(X)) = \bigoplus_{z \in {}^J W} IH^*(\mathcal{Y}_{z,J}(X), L^J_{z,w}).$$

The local systems $L^J_{z,w}$ correspond to representations of a certain subgroup $W^z_{J_z}$ of W. The subgroup $W^z_{J_z}$ is isomorphic to a product of symmetric groups.

Conjecture

The representations corresponding to $L^J_{z,w}$ are permutation representations.

Example

Fix $w = 3412 \in S_4$, we have a natural map

$$\mathcal{Y}_w(X) = \{V_\bullet; XV_1 \subset V_3, V_1 \subset XV_3\} \xrightarrow{f} \mathrm{Gr}(2,4) = \mathcal{Y}_{3412,\{1,3\}}(X).$$
$$V_1 \subset V_2 \subset V_3 \mapsto V_2.$$

This map is generically $2:1$. Let $\mathcal{U}$ be the open set of $\mathrm{Gr}(2,4)$ where the map is $2:1$, we let L be the local system given by $L = f_*(\mathbb{C}_{f^{-1}(\mathcal{U})})$. This L corresponds to the regular representation of S_2, which is seen as a subgroup of S_4 via the diagonal inclusion $S_2 \to S_2 \times S_2 \subseteq S_4$. The decomposition theorem reads

$$IH^*(\mathcal{Y}_w(X)) = IH^*(\mathcal{Y}_{3412,J}(X), L) \oplus IH^*(\mathcal{Y}_{1342,J}, \mathbb{C}[-1]) \oplus$$
$$\oplus\, IH^*(\mathcal{Y}_{3124,J}(X), \mathbb{C}[-1]) \oplus IH^*(\mathcal{Y}_{1234,J}, \mathbb{C}[-2]).$$

$J = \{n-k+1, \ldots, n-1\}$

Theorem

When $J = \{n-k+1, \ldots, n-1\}$, then $W^z_{J_z} = S_1^{n-k'} \times S_{k'}$ for some $k' = k'(z) \leq k$. In fact, k' is the largest index less or equal than k such that $z \in S_{n-k'} \times S_1^{k'}$. Moreover, if we write $z = \overline{z} \times \{1\}^{k'} \in S_{n-k'} \times S_1^{k'}$, we have that

$$\mathrm{ch}(IH^*(\mathcal{Y}_{z,J}(X), L^J_{z,w})) = \mathrm{ch}(IH^*(\mathcal{Y}_{\overline{z},\overline{J}}(X))\,\mathrm{ch}(L^J_{z,w}).$$

Recall that $L^J_{z,w}$ corresponds to a $S_{k'}$ representation.

Conjecture

If $J = \{n-k+1, \ldots, n-1\}$ and w is codominant, then $L^J_{z,w}$ is zero if z is not codominant.

Example

- $w = 23451$, $\mathbf{m} = (2,3,4,5,5)$.

$$\mathcal{Y}_w(X) = \{V_\bullet; XV_1 \subset V_2, XV_2 \subset V_3, XV_3 \subset V_4\}$$

- Forget V_4

$$\mathcal{Y}_{23451,\{4\}} = \{V_1 \subset V_2 \subset V_3; XV_1 \subset V_2, XV_2 \subset V_3\}$$
$$\mathcal{Y}_{23145,\{4\}} = \{V_1 \subset V_2 \subset V_3; XV_1 \subset V_2, XV_3 = V_3\}$$

- the map $\mathcal{Y}_w \to \mathcal{Y}_{w,\{4\}}$ is birational, an isomophism over $\mathcal{Y}_{23451,\{4\}} \setminus \mathcal{Y}_{23145,\{4\}}$ and the fibers over $\mathcal{Y}_{23145,\{4\}}$ are $\mathbb{P}^1$.
-

$$\mathrm{ch}(IH^*(\mathcal{Y}_w)) = \mathrm{ch}(IH^*(\mathcal{Y}_{w,\{4\}})) + q\,\mathrm{ch}(IH^*(\mathcal{Y}_{23145,\{4\}}))$$

Example

- 23145 splits at 3, so

$$\mathrm{ch}(IH^*(\mathcal{Y}_{23145,\{4\}})) = \mathrm{ch}(IH^*(\mathcal{Y}_{231}))h_2 = (qh_{2,1}+(q^2+q+1)h_3)h_2.$$

- Forget V_3

$$\mathcal{Y}_{23451,\{3,4\}} = \{V_1 \subset V_2; XV_1 \subset V_2\}$$
$$\mathcal{Y}_{21345,\{3,4\}} = \{V_1 \subset V_2; XV_2 = V2\}$$

- the map $\mathcal{Y}_{w,\{4\}} \to \mathcal{Y}_{w,\{3,4\}}$ is birational, and the fibers over $\mathcal{Y}_{21345,\{3,4\}}$ are $\mathbb{P}^2$.

$$\mathrm{ch}(IH^*(\mathcal{Y}_{w,\{4\}})) = \mathrm{ch}(IH^*(\mathcal{Y}_{w,\{3,4\}})) + (q^2+q)\,\mathrm{ch}(IH^*(\mathcal{Y}_{21345,\{3,4\}}))$$

- 21345 splits,

$$\mathrm{ch}(IH^*(\mathcal{Y}_{21345,\{3,4\}})) = \mathrm{ch}(IH^*(\mathcal{Y}_{21}))h_3 = (q+1)h_2h_3.$$

Example

- Forget $V2$

$$\mathcal{Y}_{23451,\{2,3,4\}} = \{V_1\} = \mathbb{P}^{n-1}$$
$$\mathcal{Y}_{12345,\{2,3,4\}} = \{V_1; XV_1 = V1\}$$

- the map $\mathcal{Y}_{w,\{3,4\}} \to \mathcal{Y}_{w,\{2,3,4\}}$ is birational, and the fibers over $\mathcal{Y}_{12345,\{2,3,4\}}$ are $\mathbb{P}^3$, so we have

$$\mathrm{ch}(IH^*(\mathcal{Y}_{w,\{3,4\}})) = \mathrm{ch}(IH^*(\mathcal{Y}_{w,\{2,3,4\}})) + (q^3+q^2+q)\,\mathrm{ch}(IH^*(\mathcal{Y}_{12345,\{$$

- We have that $\mathrm{ch}(IH^*(\mathcal{Y}_{w,\{2,3,4\}})) = [5]_q h_5$.
- 12345 splits, so

$$\mathrm{ch}(IH^*(\mathcal{Y}_{12345,\{2,3,4\}}) = \mathrm{ch}(\mathcal{Y}_1(X))h_4 = h_{4,1}.$$

Example

$$
\begin{aligned}
\mathrm{ch}(\mathcal{Y}_w(X)) =& \,\mathrm{ch}(IH^*(\mathcal{Y}_{w,\{4\}})) + q\,\mathrm{ch}(IH^*(\mathcal{Y}_{23145,\{4\}})) \\
=& \,\mathrm{ch}(IH^*(\mathcal{Y}_{w,\{3,4\}})) + (q^2+q)\,\mathrm{ch}(IH^*(\mathcal{Y}_{21345,\{3,4\}})) + \\
& + q(qh_{2,1} + (q^2+q+1)h_3)h_2 \\
=& \,\mathrm{ch}(IH^*(\mathcal{Y}_{w,\{2,3,4\}})) + (q^3+q^2+q)\,\mathrm{ch}(IH^*(\mathcal{Y}_{12345,\{2,3,4\}})) \\
& + (q^2+q)(q+1)h_{3,2} + q^2h_{2,2,1} + (q^3+q^2+q)h_{3,2} \\
=& [5]_q h_5 + (q^3+q^2+q)h_{4,1} + (2q^3+3q^2+2q)h_{3,2} \\
& + q^2h_{2,2,1}
\end{aligned}
$$

Outline

Lusztig varieties

Parabolic Lusztig varieties

Forgetting to $\mathbb{P}^{n-1}$

Forgetting to $\mathbb{P}^{n-1}$

- Consider the map $\mathcal{Y}_{\mathbf{m}}(X) \to \mathbb{P}^{n-1} = \mathrm{Gr}(1,n)$, $V_\bullet \mapsto V_1$.
- The decomposition theorem gives

$$H^*(\mathcal{Y}_{\mathbf{m}}(X)) = \bigoplus_{i=0}^{n-1} IH^*(\mathcal{H}_i, L_i).$$

- Moreover, each local system L_i is associated to a representation of S_i.
- We have that $\mathrm{ch}(IH^*(\mathcal{H}_i, L_i)) = [n-i]_q h_{n-i}\,\mathrm{ch}(L_i)$.
- Our goal: find a combinatorial interpretation for $\mathrm{ch}(L_i)$.

Definitions

- $S_{n,\mathbf{m}} := \{\sigma \in S_n; \sigma(i) \leq \mathbf{m}(i)\}$.
- $\sigma = \tau_1 \cdots \tau_k$, cycle decomposition, each cycle starts with its smaller element, and the smaller elements are written increasingly: $\sigma = (1542)(396)(78)$.
- σ^c is the permutation obtained from removing the parenthesis in the cycle decomposition. $\sigma^c = 154239678$.
- $\mathrm{wt}_{\mathbf{m}}(\sigma)$ is the number of **m**-inversions of σ^c.
- $nh_n = \sum_{i=1}^{n} h_{n-i} p_n$

Theorem (Stanley)

$$\omega(\mathrm{csf}(G_{\mathbf{m}})) = \sum_{\sigma = \tau_1 \cdots \tau_j \in S_{n,\mathbf{m}}} p_{|\tau_1|} \cdots p_{|\tau_k|}$$

Definitions

- $S_{n,\mathbf{m}} := \{\sigma \in S_n; \sigma(i) \leq \mathbf{m}(i)\}$.
- $\sigma = \tau_1 \cdots \tau_k$, cycle decomposition, each cycle starts with its smaller element, and the smaller elements are written increasingly: $\sigma = (1542)(396)(78)$.
- σ^c is the permutation obtained from removing the parenthesis in the cycle decomposition. $\sigma^c = 154239678$.
- $\mathrm{wt}_{\mathbf{m}}(\sigma)$ is the number of **m**-inversions of σ^c.
- $[n]_q h_n = \sum_{i=1}^{n} h_{n-i}\rho_n$

Theorem (A-N)

$$\omega(\mathrm{csf}_q(G_{\mathbf{m}})) = \sum_{\sigma=\tau_1\cdots\tau_j\in S_{n,\mathbf{m}}} q^{\mathrm{wt}_{\mathbf{m}}(\sigma)}\rho_{|\tau_1|}\cdots\rho_{|\tau_k|}$$

A combinatorial interpretation

Theorem

We have that $g_k(G_{\mathbf{m}}) := \omega(\mathrm{ch}(L_k))$ *(*$k = 0, \ldots, n-1$*) is equal to*

$$\sum_{\substack{\sigma=\tau_1\cdots\tau_j \in S_{n,\mathbf{m}} \\ |\tau_1| \geq n-k}} (-1)^{k-j+1} h_{|\tau_1|-n+k} p_{|\tau_2|} \cdots p_{|\tau_j|}$$

The symmetric function $g_k(G_{\mathbf{m}})$ *is of degree* k*.*

A combinatorial interpretation

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The symmetric function $g_k(G_{\mathbf{m}})$ *is of degree* k*.*

Corollaries

Corollary

We have that

$$\mathrm{csf}_q(G_{\mathbf{m}}) = \sum_{i=0}^{n-1} [n-i]_q e_{n-i} g_i(G_{\mathbf{m}}).$$

Corollary

The symmetric function $g_k(G_{\mathbf{m}})$ is Schur-positive.

Problem

Find positive formulas for the coefficients of $g_k(G_{\mathbf{m}})$ in the monomial/Schur basis.

e-positivity

Theorem (The functions g_k for the path graph)

We have that $g_k(P_n) = g_k(P_{k+1})$ for every $n > k$. We also have the following generating function

$$\sum_{k\geq 0} g_k(P_{k+1})z^k = \frac{1}{1 - q\sum_{k\geq 2}[k-1]_q e_k z^k}.$$

In particular, we have that $g_k(P_n)$ is e-positive for every $n > k$.

Conjecture

The symmetric function $g_k(G_{\mathbf{m}})$ is e-positive.

We have the following generating function for $\mathrm{csf}_q(P_k)$

$$\sum_{k\geq 0} \mathrm{csf}_q(P_k)z^k = \frac{\sum_{k\geq 0} e_k z^k}{1 - q\sum_{k\geq 2}[k-1]_q e_k z^k}$$

Theorem (Positivity of the leading coefficient (only for $q = 1$))

We have that the coefficient of e_k in the e-expansion of $g_k(\mathbf{m})$ is non-negative.

Corollary

Let $\mathbf{m}\colon [n] \to [n]$ be a Hessenberg function and let $\lambda \vdash n$ be a partition of length 2. The coefficient of e_λ in $\mathrm{csf}(G_{\mathbf{m}})$ is non-negative.

Thank you!