Generation and genericity of the group of absolutely continuous homeomorphisms of the interval

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Setting

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This talk will focus on a subgroup of H_+.

Absolute continuity

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Recall the following from a first course in measure theory:

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A function $f : [0,1] \to \mathbb{R}$ is **absolutely continuous** if for every $\epsilon > 0$, there is a $\delta > 0$ such that for every finite, pairwise disjoint collection $((a_i, b_i))_{i<n}$ of open intervals in $[0,1]$, we have

$$\sum_{i<n} b_i - a_i < \delta \implies \sum_{i<n} |f(b_i) - f(a_i)| < \epsilon.$$

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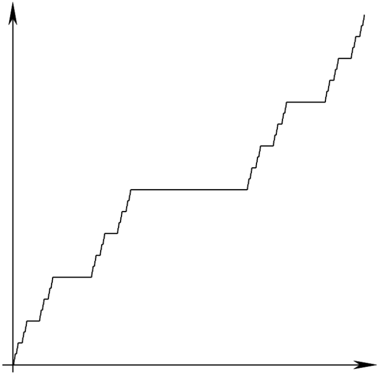
Every Lipschitz continuous function is absolutely continuous, and every absolutely continuous function has bounded variation.

Absolute continuity

Figure: The Cantor staircase is the canonical example of a non-absolutely continuous function.



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(iii) There exists a map $g \in L_1$ such that $f(x) = f(0) + \int_0^x g(t)\,\mathrm{d}t$ for all $x \in [0,1]$.

Absolutely continuous homeomorphisms

Definition

The group H_+^{AC} is the subgroup of H_+ given by:

$$H_+^{AC} := \left\{ f \in H_+ : f \text{ and } f^{-1} \text{ are absolutely continuous} \right\}.$$

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Theorem (Solecki, 1999)

The metric d_{AC} induces a Polish topology on H_+^{AC}, which is finer than the uniform convergence topology.

Other subgroups of H_+

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H_+

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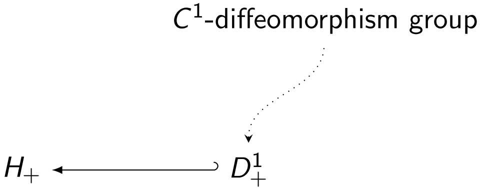

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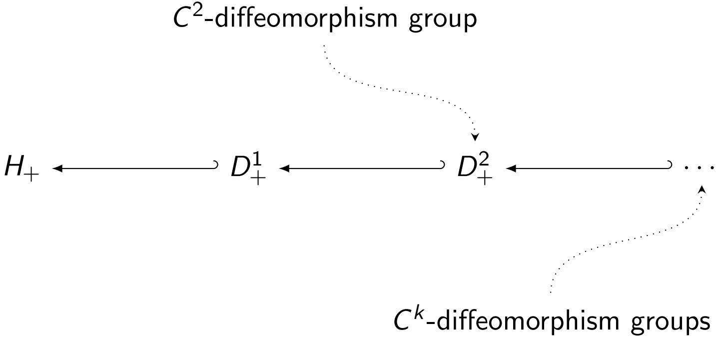

Other subgroups of H_+

$$H_+ \hookleftarrow H_+^{AC} \hookleftarrow D_+^1 \hookleftarrow D_+^2 \hookleftarrow \cdots$$

Solecki, 1999

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Solecki, 1999

Cohen, 2020

Other subgroups of H_+

topologies get finer $\longrightarrow$

$$H_+ \hookleftarrow H_+^{AC} \hookleftarrow D_+^1 \hookleftarrow D_+^{1+AC} \hookleftarrow D_+^2 \hookleftarrow D_+^{2+AC} \hookleftarrow \cdots$$

$\longleftarrow$ images are dense

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  is non-empty.
- G is **generically n-generated** if Ω_n is a dense G_δ (hence comeagre) set in G^n.
- The **topological rank** (resp. **generic rank**) is the least $n \leq \aleph_0$ for which G is topologically (resp. generically) n-generated.

Generation

topological/generic rank is non-decreasing $\longrightarrow$

$$H_+ \hookleftarrow H_+^{AC} \hookleftarrow D_+^1 \hookleftarrow D_+^{1+AC} \hookleftarrow D_+^2 \hookleftarrow D_+^{2+AC} \hookleftarrow \cdots$$

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2

e.g. Thompson's group F

Generation

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2

2

Akhmedov–Cohen, 2019

Generation

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Generic topological rank:

$$H_+ \hookleftarrow H_+^{AC} \hookleftarrow D_+^1 \hookleftarrow D_+^{1+AC} \hookleftarrow D_+^2 \hookleftarrow D_+^{2+AC} \hookleftarrow \cdots$$

H_+: 2 (topological rank), 2 (generic topological rank)

D_+^1: ≤ 10 (topological rank)

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Generation

Topological rank:

Generic topological rank:

$$H_+ \hookleftarrow H_+^{AC} \hookleftarrow D_+^1 \hookleftarrow D_+^{1+AC} \hookleftarrow D_+^2 \hookleftarrow D_+^{2+AC} \hookleftarrow \cdots$$

H_+	H_+^{AC}	D_+^1	D_+^{1+AC}	D_+^2	D_+^{2+AC}	$\cdots$
2		≤ 10		≤ 13		$(\leq 3k+7)$
2						

Akhmedov, in prep.

Generation

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H_+	H_+^{AC}	D_+^1	D_+^{1+AC}	D_+^2	D_+^{2+AC}
2	≤ 10	≤ 10	≤ 13	≤ 13	≤ 16
2					

Since topological rank non-decreases

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2	2	? →			

I., 2022

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(ii) The intervals of "positive parity" and "negative parity" are dense in this order.

Parity

Genericity

Theorem (I., 2022)

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(iii)

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(ii) The intervals of "positive parity" and "negative parity" are dense in this order;

(iii) $\mathrm{Fix}(f)$ is Lebesgue null.

Genericity

Akhmedov–Cohen, 2019

Akhmedov, in prep.

I., 2022

Topological rank:

Generic topological rank:

$$H_+ \hookleftarrow H_+^{AC} \hookleftarrow D_+^1 \hookleftarrow D_+^{1+AC} \hookleftarrow D_+^2 \hookleftarrow D_+^{2+AC} \hookleftarrow \cdots$$

	H_+	H_+^{AC}	D_+^1	D_+^{1+AC}	D_+^2	D_+^{2+AC}
Topological rank	2	2	≤ 10	≤ 13	≤ 13	≤ 16
Generic topological rank	2	2	? →			

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I., 2022

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Generic elements:	Yes	Yes					

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I., 2022

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Generic topological rank:	2	2	? $\longrightarrow$				
Generic elements:	Yes	Yes	No $\longrightarrow$				