1-D FRACTIONAL DIFFUSION: EQUIVALENT MODELS, STRUCTURE OF THE SOLUTION, NOVEL NUMERICAL SCHEME

Yulong Li

Department of Mathematics & Statistics, University of Nevada Reno

One of the models in fractional diffusion

Consider 1-D double-sided fractional diffusion advection reaction equation

$$\begin{cases} [L(u)](x) = f(x),\ x \in \Omega = (a,b), \\ u(a) = u(b) = 0, \\ [L(u)](x) := -Dk(x)(\alpha\, {}_aD_x^{-(1-\mu)} + \beta\, {}_xD_b^{-(1-\mu)})Du \\ \qquad +p(x)Du + q(x)u(x), \end{cases} \tag{1}$$

where $0 < \mu < 1$, $0 \le \alpha, \beta \le 1$, $\alpha + \beta = 1$, ${}_aD_x^{-(1-\mu)}$, ${}_xD_b^{-(1-\mu)}$ represent left- and right-sided Riemann-Liouville integrals.

Model (1) has been used in modeling anomalous diffusion processes such as in underground water, cellular cytoplasm setting, etc.

Meanwhile, some disputes arise on the wellposedness of various models and the challenge of numerical approximations. In the following, we present some recent progress, which partial solves the open questions.

Structure of solution

[1] Under natural conditions on the coefficents $k(x), p(x), q(x), f(x)$, there exists a unique true solution $u(x)$ in $\widehat{H}_0^{(1+\mu)/2}(\Omega)$ to (1). It can be decomposed into

$$\begin{aligned} u(x) = &\int_a^x \int_a^{x_1} v(t)\,dt\,dx_1 - \frac{cS}{S_1}\int_a^x \int_a^{x_1} (t-a)^p(b-t)^q\,dt\,dx_1 \\ &+ \frac{c_1 S}{S_1}(x-a)^{p+1}(b-x)^{q+1} \\ &+ C\int_a^x (t-a)^p(b-t)^q\,dt,\ x \in \Omega \end{aligned} \tag{2}$$

where $v(x)$ is a certain function in $H^*(\Omega) \subset L^1(\Omega)$, C is a certain constant, p, q are uniquely determined by

$$p + q = -(1-\mu) \quad \text{and} \quad \alpha\sin(q\pi) = \beta\sin(p\pi),$$

$$c = \Gamma(-q)\left(\alpha(b-a)^{1+p+q}\Gamma(p+1)\int_a^b (t-a)^{-q-1}(b-t)^{-p-1}\,dt\right)^{-1},$$

$$S_1 = \frac{\Gamma(-q)\int_a^b (t-a)^p(b-t)^q\,dt}{\alpha(b-a)^{1+p+q}\Gamma(p+1)\int_a^b (t-a)^{-q-1}(b-t)^{-p-1}\,dt},$$

$$c_1 = \frac{(-p-1+\mu)(-p+\mu)\Gamma(1-\mu)}{\mu(1+\mu)\beta\,\Gamma(2-\mu+p)\Gamma(q+2)}, \qquad S = \int_a^b v(t)\,dt,$$

and it is representable by

$$u(x) = {}_aD_x^{-(1+\mu)}J, \quad x \in \Omega, \text{ for a certain } J(x) \in H^*(\Omega) \subset L^1(\Omega). \tag{3}$$

Equivalent models

Based on the structure of solution, we can prove that the following models are actually equivalent:

$$\begin{cases} [L(u)](x) = f(x),\ x \in \Omega = (a,b), \\ u(a) = u(b) = 0, \\ [L(u)](x) := -Dk(x)(\alpha\, {}_aD_x^{-(1-\mu)} + \beta\, {}_xD_b^{-(1-\mu)})Du \\ \qquad +p(x)Du + q(x)u(x), \end{cases} \iff$$

$$\begin{cases} [L(u)](x) = f(x),\ x \in \Omega = (a,b), \\ u(a) = u(b) = 0, \\ [L(u)](x) := -Dk(x)D(\alpha\, {}_aD_x^{-(1-\mu)} + \beta\, {}_xD_b^{-(1-\mu)})u \\ \qquad +p(x)Du + q(x)u(x), \end{cases} \iff$$

$$\begin{cases} [L(\tilde{u})](x) = f(x),\ x \in \Omega, \\ \tilde{u}(x) = 0, x \in \mathbb{R}\backslash\Omega \\ [L(\tilde{u})](x) := -k(x)\left((\alpha-\beta)_aD_x^{(1+\mu)} + 2\beta\cos((1+\mu)\pi/2)(-\Delta)^{(1+\mu)/2}\right)\tilde{u} \\ \qquad -k'(x)\left({}_aD_x^{\mu} - 2\beta\cos(\mu\pi/2)(-\Delta)^{\mu/2}\right)\tilde{u} \\ \qquad +p(x)D\tilde{u} + q(x)\tilde{u}(x), \end{cases}$$

where $\tilde{u}(x)$ denotes the zero extension of $u(x)$ outside Ω.

Novel numerical scheme

What does the structure of solution tell us? Observe that

$$u(x) = {}_aD_x^{-(1+\mu)}\psi + C_1(x-a)^{t_1+1}(b-x)^{t_2+1} + C_2\int_a^x (t-a)^{t_1}(b-t)^{t_2}\,dt, \tag{4}$$

for $x \in (a,b)$, where ψ, C_1, C_2 are unknonwn and everything else is known. Due to the last two "bad" terms, it is usually inaccurate to make a priori assumption on the smoothness of u to allow for an optimal convergence rate in the analysis of numerical approximations.

What is the novel numerical scheme? Rather than approximating u directly, which is usually challenging and can be tough, we approximate ψ and the constants C_1, C_2, which in turn gives the approximation of u after doing back substitution.

what is the chief merit of this method? Unlike standard numerical approximations, this approach poses no necessity to directly approximate the less regular components of the solution. What is mainly left is to approximate ψ. Once approximation of ψ is available, then the constants C_1, C_2 can be calculated, and approximation of u can be directly constructed from (4). Moreover, the error of this construction is only due to the approximation of ψ, and thus is free from the nonsmoothness effects inherent in u.

One-sided case

To test the philosophy and provide for a manageable set of tasks in successfully developing, analyzing, and implementing the above approach, we first try the special case, $\beta = 0$:

$$\begin{cases} [L(u)](x) = f(x), \quad x \in \Omega = (a,b), \\ u(a) = u(b) = 0, \\ [L(u)](x) := -D(k(x)_aD_x^{-(1-\mu)}Du) + p(x)Du + q(x)u. \end{cases} \tag{5}$$

The corresponding solution is reduced to

$$u(x) = {}_aD_x^{-(1+\mu)}\psi + C_\psi(x-a)^\mu, x \in \overline{\Omega},\ C_\psi = -(b-a)^{-\mu}{}_aD_x^{-(1+\mu)}\psi\big|_{x=b}.$$

Algorithm: instead directly approximating solution u, we approximate the unknown ψ and C_ψ

Step 1: convert the fractional differential equation into integral equation

$$\psi + \mathcal{I}\psi = F(x) - C_\psi G(x), \ \text{ a.e. in } \overline{\Omega}, \tag{6}$$

where

$$\begin{aligned} [\mathcal{I}\psi](x) &:= -\frac{1}{k(x)}\left(p(x)\,{}_aD_x^{-\mu}\psi + q(x)\,{}_aD_x^{-(1+\mu)}\psi - k'(x)\,{}_aD_x^{-1}\psi\right), \\ G(x) &:= -\frac{1}{k(x)}\left(\mu\, p(x)(x-a)^{\mu-1} + q(x)(x-a)^\mu - \Gamma(\mu+1)k'(x)\right), \\ F(x) &:= -f/k. \end{aligned}$$

Step 2: Use Two-Step Method to approximate ψ_F and ψ_G that are governed by

$$\begin{cases} \psi_F + \mathcal{I}\psi_F = F(x),\ x \in \Omega, \\ \psi_G + \mathcal{I}\psi_G = G(x),\ x \in \Omega. \end{cases}$$

Step 3:

$$\psi = \psi_F - C_\psi\psi_G \Rightarrow \begin{cases} \text{calculate } C_\psi \\ \text{by } u(b) = 0. \end{cases}$$

Step 4: Do error analysis and related convergence analysis. (Numerical experiments can be found in [2])

References

[1] Yulong Li. "On the decomposition of solutions: from fractional diffusion to fractional Laplacian". In: *Fract. Calc. Appl. Anal.* 24.5 (2021), pp. 1571–1600. ISSN: 1311-0454.

[2] Yulong Li and Ginting Victor. "On the Dirichlet BVP of fractional diffusion advection reaction equation in bounded interval: structure of solution, integral equation and approximation". In: *(submitted)* (2022).