# Fractional powers of first order differential operators and inverse measures

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# Reference

- Joint work with Martín Mazzitelli (Instituto Balseiro, Argentina) and José L. Torrea (Universidad Autónoma de Madrid), arXiv 2022.

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- Hermite polynomials form an orthogonal basis of $L^2(\mathbb{R}, d\gamma_1)$
- Gaussian harmonic analysis: $d\gamma_1$ is nondoubling and non-Ahlfors regular, so classical CZ theory on metric measure spaces does not directly apply.

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- Orthonormal basis of $L^2(\mathbb{R}, d\gamma_{-1})$? Not polynomials anymore!
- *Inverse* Gaussian harmonic analysis: similar obstructions.

# Inverse Gaussian measure in geometry

A hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ is a self-expander if

$$\mathbf{H}_\Sigma = \frac{1}{2}\mathbf{x}^\perp$$

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A self-expander $M$ is a critical point of the weighted volume functional

$$F(M) = \int_M e^{|x|^2/4}\, d\mathcal{H}^{n-1}$$

**Reference.** T. Ilmanen, *Lectures on mean curvature flow and related equations* (1995)

# Fractional derivatives and integrals

For $u : \mathbb{R} \to \mathbb{R}$, let

$$D_{left} u(x) = \lim_{t \to 0^+} \frac{u(x) - u(x-t)}{t} = - \lim_{t \to 0^+} \frac{e^{-tD_{left}} u(x) - u(x)}{t}$$

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For $0 < \alpha < 1$,

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is the **Weyl fractional integral**.

# Marchaud fractional derivative: PDE results

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- Using
$$\int_{\mathbb{R}} ((D_{left})^{\alpha} u) v \, dx = \int_{\mathbb{R}} u((D_{right})^{\alpha} v) \, dx$$
and **one-sided test functions**, we can define $(D_{left})^{\alpha} u$ in the sense of distributions.

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- **Maximum principle.** If $(D_{left})^\alpha u \leq 0$ in $(0, T]$ and $u \leq 0$ in $(-\infty, 0]$ then $\sup_{(-\infty,T]} u = \sup_{(-\infty,0]} u$.

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- **Extension problem.** If $U = U(x, y)$ solves
$$\begin{cases} -D_{left} U + \frac{1-2\alpha}{y} U_y + U_{yy} = 0 & \text{for } x \in \mathbb{R}, \; y > 0 \\ U(x,0) = u(x) & \text{on } \mathbb{R} \end{cases}$$
then $-d_{\alpha} y^{1-2\alpha} U_y(x,y)\big|_{y=0} = (D_{left})^{\alpha} u(x)$.

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then $-d_{\alpha} y^{1-2\alpha} U_y(x,y)\big|_{y=0} = (D_{left})^{\alpha} u(x)$.
- **Harnack inequality.** If $u \geq 0$ in $\mathbb{R}$, and $(D_{left})^{\alpha} u = 0$ in $(0,1)$, then
$$\sup_{(1/4,1/3)} u \leq C \inf_{(1/2,3/4)} u.$$

# Marchaud fractional derivative: real analysis results

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- **One-sided fractional Sobolev spaces.** For $1 \leq p < \infty$ and $\omega \in A_p^-(\mathbb{R})$ (one-sided Sawyer weight)

$$W^{\alpha,p}(\omega^p) = \{u = (D_{left})^{-\alpha} f : f \in L^p(\omega^p)\}$$

Characterizations with left fractional derivatives.

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  Characterizations with left fractional derivatives.

- **Bourgain–Brezis–Mironescu-type result.** For

$$W^{1,p}(\omega) = \{u \in L^p(\omega) : D_{left} u \in L^p(\omega)\}$$

  we have

$$\lim_{\alpha \to 1} (D_{left})^\alpha u = D_{left} u \qquad \text{in } L^p(\omega),\ 1 < p < \infty,\ \text{and a.e.}$$

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- **Fundamental Theorem of Fractional Calculus.**

$$u(x) = \lim_{\varepsilon \to 0} (D_{left})^{\alpha}_{\varepsilon} (D_{left})^{-\alpha} u(x) \qquad \text{in } L^p(\omega^p) \text{ and a.e.}$$

# First order differential operators

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## Lemma (Semigroup)

$$e^{-t\mathfrak{D}_{left,a}}u(x) = \mathcal{E}(x)e^{-tD_{left}}(\mathcal{E}^{-1}u)(x)$$

# Fractional powers of first order differential operators

**Positive fractional power** is a *modulation* of Marchaud:

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(\mathfrak{D}_{left,a})^\alpha u(x) &= \frac{1}{\Gamma(-\alpha)} \int_0^\infty \left(e^{-t\mathfrak{D}_{left,a}} u(x) - u(x)\right) \frac{dt}{t^{1+\alpha}} \\
&= c_\alpha \mathcal{E}(x) \int_{-\infty}^x \frac{(\mathcal{E}^{-1}u)(x) - (\mathcal{E}^{-1}u)(t)}{(x-t)^{1+\alpha}}\, dt \\
&= \mathcal{E}(x)(D_{left})^\alpha(\mathcal{E}^{-1}u)(x)
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**Negative fractional power** is a *modulation* of Weyl:

$$(\mathfrak{D}_{left,a})^{-\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-t\mathfrak{D}_{left,a}}u(x)\,\frac{dt}{t^{1-\alpha}}$$
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- We can **transfer** results from $(D_{left})^{\pm\alpha}$ to $(\mathfrak{D}_{left,a})^{\pm\alpha}$ using modulation (semigroup, distributions, maximum principle, extension, BBM-type results, etc)

# Sobolev spaces

Recall that, for $\omega \in A_p^-(\mathbb{R})$ and $a \equiv 0$,

$$W^{1,p}(\omega) = \left\{ u \in L^p(\omega) : D_{left} u \in L^p(\omega) \right\}$$

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Since, for any $a(x)$,

$$\mathcal{E}(x)^{-1} \mathfrak{D}_{left,a} u = D_{left}(\mathcal{E}^{-1} u)(x)$$

then the correct definition of Sobolev space associated with $\mathfrak{D}_{left,a}$ is

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Similarly,

$$W_a^{\alpha,p}(\omega^p) = \{u = (\mathfrak{D}_{left,a})^{-\alpha} f : f \in L^p(\mathcal{E}^{-p}\omega^p)\}$$

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**Conclusion.** The natural $L^p$ space for analysis of $(\mathfrak{D}_{left,a})^{\pm\alpha}$ is

$$L^p(\mathcal{E}^{-p}) = \{ u : \mathbb{R} \to \mathbb{R} : \mathcal{E}^{-1} u \in L^p(\mathbb{R}) \}$$

# An important example

When $a(x) = x$,

$$\mathfrak{D}_{left,a} u = D_{left} u + xu$$

- This is the natural *derivative* in harmonic analysis of Hermite expansions:

$$-\frac{d^2}{dx^2} + x\frac{d}{dx} = (D_{left} + x)(D_{right})$$

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By taking $x_0 = 0$,

$$\mathcal{E}(x)^{-1} = \exp\left(\int_0^x a(y)\,dy\right) = e^{x^2/2}$$

and we end up with the **inverse Gaussian space**

$$L^2(\mathcal{E}^{-2}) = \{u : \mathbb{R} \to \mathbb{R} : e^{x^2/2} u \in L^2(\mathbb{R})\} = L^2(\mathbb{R}, d\gamma_{-1})$$

# Eigenpolynomials for inverse Gaussian

**Theorem (Mazzitelli–S.–Torrea, 2022)**

*The polynomials given by the Rodrigues formula*

$$\mathcal{H}(x) = (-1)^n e^{-x^2} \frac{d^n}{dx^n}(e^{x^2}) \qquad n \geq 0$$

*are eigenfunctions of* $\mathcal{L} = \frac{d^2}{dx^2} + 2x\frac{d}{dx}$ *with* $\mathcal{L}\mathcal{H}_n = 2n\mathcal{H}_n$.

*Generating function formula:* $e^{t^2-2xt} = \sum_{n=0}^{\infty} \mathcal{H}_n(x)\frac{t^n}{n!}$

*Three-term recurrence relation:* $\mathcal{H}_{n+1}(x) + 2x\mathcal{H}_n(x) - 2n\mathcal{H}_{n-1} = 0 \quad n \geq 1.$

*There is a unique moment functional* $\mathcal{F}$ *such that*

$$\mathcal{F}(1) = 1, \quad \mathcal{F}(\mathcal{H}_n\mathcal{H}_m) = 0 \quad \text{and} \quad \mathcal{F}(\mathcal{H}_n^2) \neq 0.$$

# Eigenpolynomials for inverse Gaussian

**Theorem (Mazzitelli–S.–Torrea, 2022)**

*The polynomials given by the Rodrigues formula*

$$\mathcal{H}(x) = (-1)^n e^{-x^2} \frac{d^n}{dx^n}(e^{x^2}) \qquad n \geq 0$$

*are eigenfunctions of* $\mathcal{L} = \frac{d^2}{dx^2} + 2x\frac{d}{dx}$ *with* $\mathcal{L}\mathcal{H}_n = 2n\mathcal{H}_n$.

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- Notice that $\mathcal{H}(x)$ are not *orthogonal* with respect to $d\gamma_{-1}$, but with respect to a moment functional $\mathcal{F}$.

# Orthogonal basis for inverse Gaussian

## Theorem

*Consider the classical Hermite polynomials given by the Rodrigues formula*

$$H_n(x) = (-1)^n e^{x^2} \frac{d}{dx}(e^{-x^2}) \qquad n \geq 0.$$

*Then*

$$H_n^*(x) = e^{-x^2} H_n(x)$$

*forms an orthogonal basis of* $L^2(\mathbb{R}, d\gamma_{-1})$*. Moreover,*

$$\mathcal{L}H_n^* = -(2n+2)H_n^*.$$

## Remark

*We also prove* $L^2(\mathbb{R}, d\gamma_{-1})$ *boundedness of singular integrals associated to* $\mathcal{L} = \frac{d^2}{dx^2} + 2x\frac{d}{dx}$ *(maximal semigroup operator, Riesz transform* $\frac{d}{dx}\mathcal{L}^{-1/2}$*, Littlewood–Paley square functions). The idea is to conjugate with the corresponding operators related to* $L = -\frac{d^2}{dx^2} + 2x\frac{d}{dx}$*.*

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choose $x_0 = 1$ and compute

$$\mathcal{E}^{-1}(x) = \exp\left[-\int_1^x \left(\frac{\alpha}{y} - 1\right) dy\right] = \frac{1}{e} x^{-\alpha} e^x.$$

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- We obtain boundedness of singular integrals in $L^2$ of the inverse measures.

Thank you for your attention!