Sobolev Estimates for Fractional PDEs

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BIRS, Recent Progress in Kinetic and Integro-Differential Equations

Abstract

I will discuss some recent results on Sobolev estimates for fractional elliptic and parabolic equations with or without weights. We considered equations with time fractional derivatives of the Caputo type, or with nonlocal derivatives in the space variables (eg. the fractional Laplacian), or both.

This is based on joint work with Doyoon Kim (Korea University), Yanze Liu (Brown), and Pilgyu Jung (Korea University).

Classical Calderon–Zygmund estimate

Consider elliptic equations in non-divergence form:

$$Lu := a^{ij}D_{ij}u + b^i D_i u + cu = f \quad \text{in } \Omega \subset \mathbb{R}^d.$$

Ellipticity and boundedness assumptions on coefficients:

$$\delta|\xi|^2 \le a^{ij}\xi^i\xi^j \le \delta^{-1}|\xi|^2, \quad |b^i|, |c| \le K, \quad c \le 0.$$

Classical Calderon–Zygmund estimate (provided that a^{ij} are sufficiently regular, say unif. continuous):

$$\|D^2 u\|_{L_p} \le N\|f\|_{L_p}.$$

Similar results hold for parabolic equations as well as for divergence form equations.

Classical Calderon–Zygmund estimate

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A class of regular coefficients: *VMO* coefficients

- Chiarenza–Frasca–Longo (1991, 1993): W_p^2 estimate for nondivergence form elliptic equations with bounded VMO leading coefficients.
- Di Fazio (1996): W_p^1 estimate for divergence form elliptic equations with bounded VMO leading coefficients.
- Bramanti and Cerutti (1993): nondivergence form parabolic equations with bounded VMO coefficients.
  (proofs based on the Calderón-Zygmund theorem and the Coifman-Rochberg-Weiss commutator theorem.)
  Recall: A function f is VMO if

$$\fint_{B_r(x_0)} |f - (f)_{B_r(x_0)}|\, dx \to 0$$

as $r \to 0$ uniformly in x_0.
Drawback: Need smoothness of the fundamental solutions.

The mean oscillation method

Krylov (2005): $W^{1,2}_p(\mathbb{R}^d)$ solvability of both divergence and nondivergence form parabolic equations with a^{ij} in VMO_x.
Krylov (2007): mixed-norm estimates in $L^t_q(L^x_p)$ for $q \geq p$ (also called maximal regularity estimate).

Idea: Prove pointwise estimate of sharp functions of Du (or D^2u), and use the Fefferman–Stein theorem on sharp functions and the Hardy–Littlewood maximal function theorem, i.e.,

$$\|\mathcal{M}f\|_{L_p} \leq N\|f\|_{L_p}, \quad \|f\|_{L_p} \leq N\|f^\#\|_{L_p}.$$

In some sense, the proof is based on an interpolation between $C^{2,\alpha}$*-estimate and* W^2_2*-estimate, and a perturbation argument.*

Subsequent work (recent)

- D.–Gallarati (2016): higher-order elliptic and parabolic equations on the half-space or domains with boundary conditions satisfying the Lopatinskii-Shapiro condition (also known as "complementing condition" in Agmon-Douglis-Nirenberg).
- D.–Kim (2018): weighted and mixed-norm estimates several types of linear equations with A_p weights. By the extrapolation theorem, we removed the condition $q \geq p$ in the $L_q(L_p)$ estimate for non-divergence form parabolic equations.
- D.–Krylov (2019): weighted and mixed-norm Sobolev estimates for **fully nonlinear** elliptic and parabolic equations in the whole or half space under a relaxed convexity condition.

Level set method

- Caffarelli and Peral (1998): W_p^1 estimates for linear and quasilinear elliptic equations in divergence form with continuous coefficients, as well as the elliptic homogenization problem.

  Idea: First get the Lipschitz estimate for any weak solution u to equations with constant coefficients:

  $$\|Du\|_{L_\infty(B_{1/2})} \le N\|Du\|_{L_2(B_1)}.$$

  Then compare the solution to equations with variable coefficients to the solution to equations with constant coefficient.
  Finally, use a "crawling of ink" lemma originally due to Safonov and Krylov (1980) to estimate the level sets of $\mathcal{M}(|Du|^2)$.

Level set method (cont'd)

- Byun and Wang (2005): elliptic equations with VMO coefficients in Reifenberg flat domains.
- Sheng (2007): the L_p boundary value problems in Lipschitz domains.
- Choi, D., and Li (2018, 2021): elliptic and parabolic equations with mixed homogeneous Dirichlet and conormal boundary conditions in Reifenberg flat domains with Reifenberg flat separations.
- D. and Li (2019, 2021): Laplace and heat equations with mixed non-homogeneous Dirichlet and conormal boundary conditions in Reifenberg flat and Lipschitz domains with "close to Lipschitz" separations.

Caputo fractional derivatives

The Caputo fractional time derivative: for $\alpha \in (0,1)$,

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \left[u(s) - u(0)\right] ds,$$

which was introduced by Caputo in his 1967 paper, and has been used to model fractional (and anomalous) diffusion in plasma turbulence.

It can be considered as a derivative taking the "memory effect" (or "delay effect") into consideration.

Caputo fractional derivatives (cont'ed)

When $\alpha \in (0,1)$, if $u(0) = 0$, then

$$\partial_t^\alpha u = I^{1-\alpha} u_t = \partial_t (I^{1-\alpha} u),$$

where

$$I^{1-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-\alpha} u(s)\, ds.$$

See Hardy–Littlwood (1928).

For $\alpha \in (1,2)$,

$$\partial_t^\alpha u = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} (\partial_t^2 u)(s)\, ds.$$

When $u(0) = u_t(0) = 0$,

$$\partial_t^\alpha u = \partial_t^2 I^{2-\alpha} u = \partial_t^{\alpha-1}(u_t).$$

Equations with fractional time derivatives

Consider evolutionary equations with fractional time derivatives:

$$-\partial_t^\alpha u + a^{ij} D_{ij} u + b^i D_i u + cu = f,$$
$$-\partial_t^\alpha u + D_i(a^{ij} D_j u) + b^i D_i u + D_i(\hat{b}^i u) + cu = \operatorname{div} f.$$

- $\alpha \in (0,1)$: fractional parabolic (sub-diffusive) equations (particle sticking and trapping effects).
- $\alpha \in (1,2)$: fractional wave (or diffusion-wave) equations (wave propagating in viscoelastic media).

They are also related to non-Markovian diffusion processes with a memory effect. These equations have attracted much attention in recent years.

Hölder estimates when $\alpha \in (0,1)$:

- Zacher (2013): De Giorgi–Nash–Moser type estimates.
- Allen-Caffarelli-Vasseur (2016): De Giorgi–Nash–Moser estimates for equations with fractional operators in both t and x.

$L_q(L_p)$ estimates

- Clément-Prüss (1992): $a^{ij} = \delta_{ij}$, $\alpha \in (0,1)$, and $2/(\alpha q) + d/p < 1$ (for certain parabolic Volterra equations) by using a semi-group and operator theoretical approach.
- Zacher (2005): a^{ij} are uniformly continuous, $p = q > 1$, and

$$\alpha \in (0,1) \setminus \{2/(2p-1) - 1, 2/(p-1) - 1, 1/p, 3/(2p-1)\}.$$

  See also Zacher (2003) for equations with nonhomogeneous initial and boundary conditions and under the condition that $\lim_{|x| \to \infty} a^{ij}$ exists.
- Sakamoto-Yamamoto (2011): L_2 estimates for divergence type equations with C^1 coefficients.
- Da Prato-Iannelli, Kunstmann-Weis, Clément-Londen-Simonett, ...

$L_q(L_p)$ estimates (cont'ed)

Consider non-divergence form equations with fractional time derivatives

$$\begin{cases} -\partial_t^\alpha u + a^{ij} D_{ij} u + b^i D_i u + cu = f \quad \text{in } (0, T) \times \mathbb{R}^d, \\ u(0, \cdot) = 0 \text{ when } \alpha \in (0, 1), \quad u(0, \cdot) = u_t(0, \cdot) = 0 \text{ when } \alpha \in (1, 2). \end{cases}$$

Theorem (Kim-Kim-Lim 2017)

Let $\alpha \in (0, 2)$. If a^{ij} are uniformly continuous in x, piecewise continuous t, and $\lim_{|x|\to\infty} a^{ij}$ exists. Then the $L_q(L_p)$ estimate and solvability hold.

Proof: A representation formula for solution to the time fractional heat operator $-\partial_t^\alpha u + \Delta u$ using the Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}$$

together with a perturbation argument.

L_p estimates with VMO$_x$ coefficients when $\alpha \in (0,1)$

Theorem (D. and Doyoon Kim 2018, Adv. Math.)

Let $\alpha \in (0,1)$, $T \in (0,\infty)$, and $p \in (1,\infty)$. If a^{ij} are VMO in x and measurable in t, then the L_p (unmixed) estimate and solvability hold.

Proof: a level set argument without using any kernel estimates.

Main difficulty: hard to get local L_∞ estimates of the Hessian of solutions to locally homogeneous equations with coefficients depending only on t:

$$W_2^\alpha(\mathbb{R}) \hookrightarrow L_\infty(\mathbb{R}) \quad \text{when } \alpha > 1/2.$$

Scheme of the proof

- Using the localized L_2 estimate and the Sobolev type embedding results, for homogeneous equations in a parabolic cylinder, we have $D^2u \in L_{p_1}$ for some $p_1 > 2$ (instead of L_∞).
- By a modified level set argument, we proved the L_p estimate and solvability for non-homogeneous equations when $p \in (2, p_1)$.
- Localize the L_p estimate and use the Sobolev type embedding again to get $D^2u \in L_{p_2}$ for some $p_2 > p_1$ for homogeneous equations.
- Iterating this procedure gives the L_p estimate and solvability for any $p \in (2, \infty)$.
- For $p \in (1, 2)$, use a duality argument. To treat VMO_x coefficients, we apply a perturbation argument in the level set argument.

Weighted estimates with Muckenhoupt (A_p) weights

For any $p \in (1, \infty)$, recall that $A_p = A_p(\mathbb{R}^d)$ is the collection of all nonnegative functions ω on $\mathbb{R}^d$ such that

$$[\omega]_{A_p} := \sup_{x_0 \in \mathbb{R}^d, r>0} \left(\fint_{B_r(x_0)} \omega(x) \right) \left(\fint_{B_r(x_0)} \omega^{-\frac{1}{p-1}}(x) \right)^{p-1} < \infty.$$

By Hölder's inequality,

$$A_p \subset A_q, \quad 1 \le [\omega]_{A_q} \le [\omega]_{A_p}, \quad 1 < p < q < \infty.$$

The maximal operator is bounded in weighted L_p spaces with A_p weights.

Define the weighted mixed norm:

$$\|f\|_{L_{q,\omega_2}(L_{p,\omega_1})} = \left(\int \left(\int |f|^p \omega_1 \, dx \right)^{q/p} \omega_2 \, dt \right)^{1/q}$$

Weighted estimates with A_p weights

Theorem (Han-Kim-Park 2020)

Let $\alpha \in (0,2)$, $a^{ij} = \delta_{ij}$ (or constant), $\omega_1 \in A_p(\mathbb{R}^d)$, and $\omega_2 \in A_q(\mathbb{R})$. Then the $L_{q,\omega_2}(L_{p,\omega_1})$ estimate and solvability hold.

Proof: Establishing the mean-oscillation estimate using the kernel estimates in Kim-Kim-Lim (2017) and Kim-Lim (2016), and the weighted mixed-norm sharp function theorem (D.-Kim (2018)).

Remark (Variable coefficients?)

The above result can be extended to the case when $\omega_2 = 1$ and a^{ij} are uniformly continuous in x and piecewise continuous in t. When $p \neq q$, require $\lim_{|x|\to\infty} a^{ij}$ exists.

Estimates with A_p weights and VMO_x coefficients

With Doyoon Kim, we extended the above result to equations with VMO_x coefficients when $\alpha \in (0, 1)$.

Theorem (D.-Doyoon Kim 2020, Adv. Math.)

Let $\alpha \in (0, 1)$, a^{ij} be VMO in x and measurable in t, $\omega_1 \in A_p(\mathbb{R}^d)$, and $\omega_2 \in A_q(\mathbb{R})$. Then the $L_{q,\omega_2}(L_{p,\omega_1})$ estimate and solvability hold.

Again, our proof is kernel free. However, the iteration argument mentioned above does not work in this case.

New ideas in the proof

- We applied the mean oscillation argument by establishing a Hölder estimate of $D^2 v$, where v is a solution to the homogeneous equation.
- To get the Hölder estimate of $D^2 v$, a bootstrap argument is used based on the (unmixed) L_p estimate.
- To overcome the difficulty from the nonlocal effect in time, our strategy is to consider infinite cylinders $(-\infty, t_0) \times B_r(x_0)$ instead of the usual parabolic cylinders, and apply the Hardy–Littlewood maximal function for strong maximal functions.

Two key estimates

Lemma (Estimates for homogeneous equations)

Let $1 < p_0 < p < \infty$, $t_0 \in (0, \infty)$, *and* v *satisfy* $\partial_t^\alpha v - a^{ij}(t) D_{ij} v = 0$ *in* $(-\infty, t_0) \times B_r$, $r > 0$. *Then*

$$\left(|D^2 v|^p\right)^{1/p}_{Q_{r/2}(t_0,0)} \le N \sum_{j=1}^{\infty} j^{-(1+\alpha)} \left(|D^2 v|^{p_0}\right)^{1/p_0}_{Q_r(t_0-(j-1)r^{2/\alpha},0)},$$

$$[D^2 v]_{C^{\sigma\alpha/2,\sigma}(Q_{r/2}(t_0,0))} \le N r^{-\sigma} \sum_{j=1}^{\infty} j^{-(1+\alpha)} \left(|D^2 v|^{p_0}\right)^{1/p_0}_{Q_r(t_0-(j-1)r^{2/\alpha},0)}.$$

Two key estimates (cont'ed)

Lemma (Estimates for non-homogeneous equations)

Let $p_0 \in (1, \infty)$*,* $f \in L_{p_0}((-\infty, t_0) \times B_1)$*, and* w *be a weak solution to* $\partial_t^\alpha w - a^{ij}(t) D_{ij} w = f$ *in* $(-\infty, t_0) \times B_1$ *with the zero boundary condition on* $(-\infty, t_0) \times B_1$*. Then*

$$\left(|D^2 w|^{p_0}\right)^{1/p_0}_{Q_{1/2}(t_0,0)} \le \sum_{k=0}^{\infty} c_k \left(|f|^{p_0}\right)^{1/p_0}_{(s_{k+1}, s_k) \times B_1},$$

where $s_k = t_0 - 2^k + 1$ *and*

$$\sum_{k=0}^{\infty} c_k \le N = N(d, \delta, \alpha, p_0).$$

Fractional wave equations with VMO coefficients

A natural question: When $\alpha \in (1,2)$, can we prove the L_p estimate and solvability for equations with VMO or VMO_x coefficients?

Why is this different? In this case, even the L_2 estimate is not immediate. When $\alpha \in (0,1]$,

$$\int_0^T (\partial_t^\alpha u)u\,dt \ge \frac{1}{2}\int_0^T \partial_t^\alpha u^2\,dt$$
$$= \frac{1}{2}\int_0^T \partial_t(I^{1-\alpha}u^2)\,dt = \frac{1}{2}(I^{1-\alpha}u^2)(T) \ge 0.$$

No longer the case when $\alpha \in (1,2)$.

Fractional wave equations with VMO coefficients (cont'ed)

We addressed this question in the VMO case.

Theorem (D.–Yanze Liu, 2021)

Let $\alpha \in (1,2)$, a^{ij} *be VMO in* (t,x), $\omega_1 \in A_p(\mathbb{R}^d)$, *and* $\omega_2 \in A_q(\mathbb{R})$. *Then the* $L_{q,\omega_2}(L_{p,\omega_1})$ *estimate and solvability hold.*

Proof: We followed the strategy in D.–Kim (2020) by establishing mean oscillation estimates. The proof is more involved when $\alpha \in (1,2)$.
To estimate v, we applied (unmixed) L_p estimate due to Kim-Kim-Lim.
To estimate w, we need to solve an equation in a bounded cylindrical domain (to find at least a weak solution). For this, we also used the method of eigenfunction expansions and the Mittag-Leffler function.

In the same paper, we also considered divergence form equations as well as equations in the half space and domains.

Equations with nonlocal derivatives in x

Consider the following non-local elliptic equation (in nondivergence form) associated with pure jump Lévy process:

$$Lu = \int_{\mathbb{R}^d} \Big(u(x+y) - u(x) - y \cdot \nabla u(x) \chi^{(\sigma)}(y) \Big) K(x,y)\, dy,$$

where $\sigma \in (0,2)$ and

$$\chi^{(\sigma)} \equiv 0 \quad \text{for } \sigma \in (0,1), \quad \chi^{(1)} = 1_{y \in B_1}, \quad \chi^{(\sigma)} \equiv 1 \quad \text{for } \sigma \in (1,2).$$

Models in physics, engineering, and finance that involve long-range interactions.

Example: If $K(x,y) = c^{-1}|y|^{-d-\sigma}$ with $c = c(d,\sigma) > 0$ and $\sigma \in (0,2)$, we get the fractional Laplace operator $-(-\Delta)^{\sigma/2}$:

$$\widehat{(-\Delta)^{\sigma/2} f} = |\xi|^\sigma \hat{f}.$$

Symmetry and ellipticity condition

In the case that K is symmetric, i.e., $K(x,y)=K(x,-y)$,

$$Lu=\frac{1}{2}\int_{\mathbb{R}^d}(u(x+y)+u(x-y)-2u(x))\,K(x,y)\,dy.$$

Assume that $K(x,y)=a(x,y)|y|^{-d-\sigma}$.
When $\sigma=1$, we impose a cancellation condition: $\int_{\partial B_r} ya(x,y)\,dS_r(y)=0$ for any $r\in(0,\infty)$.
Ellipticity condition: $(2-\sigma)\delta\le a(x,y)\le(2-\sigma)\delta^{-1}$. The order of L is σ.
A weaker ellipticity condition: a is bounded and $a(x,y)\ge\tilde{a}(y/|y|)\ge 0$, where

$$\int_{S^{d-1}}|\omega\cdot\xi|^{\sigma}\tilde{a}(\omega)\,d\omega\ge\delta>0\quad\forall\xi\in S^{d-1}.$$

This condition has applications to linearized Boltzmann equations with the specular reflexive boundary condition a forthcoming work with Yan Guo and Timur Yastrzhembskiy.

We are interested in L_p estimates for such operators:
if $\lambda > 0$ and $f \in L_p(\mathbb{R}^d)$, $1 < p < \infty$, there exists a unique solution $u \in H_p^\sigma(\mathbb{R}^d)$ to the equation

$$Lu - \lambda u = f,$$

and u satisfies $\|u\|_{H_p^\sigma(\mathbb{R}^d)} \le N(d, \sigma, \delta, p)\|f\|_{L_p(\mathbb{R}^d)}$.
Here $H_p^\sigma = (1 - \Delta)^{-\sigma/2} L_p$ is the Bessel potential space.

- $L = -(-\Delta)^{\sigma/2}$, the classical theory for pseudo-differential operators.
- In general, if the symbol of the operator is smooth and its derivatives satisfy appropriate decays, the L_p-solvability follows from the classical Fourier multiplier theorems.

Known results

When $K(x,y) = K(y)$, the symbol of L is given by

$$m(\xi) = \int_{\mathbb{R}^d} \left(e^{iy\cdot\xi} - 1 - iy\cdot\xi\chi^{(\sigma)}(y) \right) K(y)\, dy,$$

which generally lacks sufficient differentiability to apply the classical multiplier theorems.

- Mikulevicius–Pragarauskas (1992): If $K(y) = a(y)/|y|^{d+\sigma}$ and $a(y)$ is homogeneous of order zero and sufficiently smooth, the fundamental solution to the non-local equation can be well analyzed, so that one can apply the Marcinkiewicz theorem to get a desired L_p-estimate.
- In the symmetric case, the L_p-estimate was obtained by Bañuelos–Bogdan (07) using probabilistic methods (best constants).

Known results (cont'ed)

- D.-Kim (2011): $a(x,y) = a(y)$ measurable with no symmetry condition. We also proved the continuity of the operator L from H_p^σ to L_p.
- X. Zhang (2013): the maximal $L_{p,q}$ estimate for non-local parabolic equations.
- Mikulevicius–Pragarauskas (2014): non-local parabolic equations with $a(t,x,y)$, where a is C^β in x and $\beta > d/p$. They proved that under this condition, the operator $L : H_p^\sigma \to L_p$ is continuous.

Question: Is it possible to relax the regularity condition of a on x: eg. C^β for small β, Dini continuous, uniform continuous, or even VMO?

Theorem (D.-Kim-Jung, 2021)

The operator L is continuous from H^σ_p to L_p provided that $a(x,y)$ is C^β in x for any $\beta > 0$. Consequently, we have the Sobolev estimate and solvability for the nonlocal elliptic and parabolic equations with the operator L.

In fact, we proved that L is continuous from $H^\sigma_{p,w}$ to $L_{p,w}$ for any $w \in A_p$. The proof is based on weighted estimates first for $p > d/\beta$ with A_p weights, and then uses the extrapolation theorem to obtain the weighted estimate for any $p \in (1,\infty)$.
We also used some ideas in Mikulevicius–Pragarauskas (2014).

It remains open whether one can further relax the condition on $a(x,y)$.

For nonlocal operators divergence form, see the recent work of Fall-Mengesha-Schikorra-Yeepo (2021), and S. Nowak (2021), where VMO type coefficients were treated.

Equations with space-time nonlocal operators

Recently, we also considered fractional parabolic equations with space-time non-local operators $\partial_t^\alpha u - Lu = f$ in $(0, T) \times \mathbb{R}^d$.

Theorem (D.-Yanze Liu, 2021)

Assume that $a(t, x, y)$ is C^β in x for some $\beta > 0$. Then $\partial_t^\alpha - L$ is a continuous operator from $\mathbb{H}_{p,0}^{\alpha,\sigma}((0, T) \times \mathbb{R}^d)$ to $L_p((0, T) \times \mathbb{R}^d)$. Moreover, for any $u \in \mathbb{H}_{p,0}^{\alpha,\sigma}((0, T) \times \mathbb{R}^d)$ satisfying

$$\partial_t^\alpha u - Lu = f \quad \text{in} \quad (0, T) \times \mathbb{R}^d,$$

we have

$$\|u\|_{\mathbb{H}_p^{\alpha,\sigma}((0,T)\times\mathbb{R}^d)} \le N \|f\|_{L_p((0,T)\times\mathbb{R}^d)} \,.$$

When $\alpha = 1$, we also obtained weighted mixed-norm estimates.

Thank you for your attention!