Order Reconfiguration under Width Constraints

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BIRS Reconfiguration Workshop 22w5090 May 10th, 2022
also see: MFCS 2021

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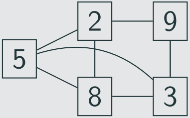
[2] University of Trier, Germany

Overview

Graph isomorphism

G_1 :



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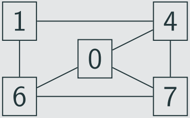
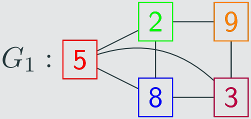
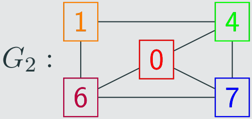


Overview

Graph isomorphism





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Graph isomorphism

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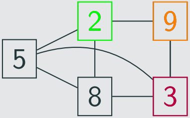

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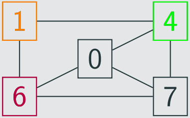
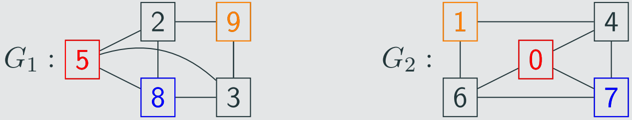


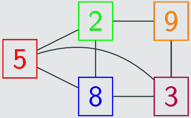
Overview

Graph isomorphism



Graph isomorphism

G_1 :



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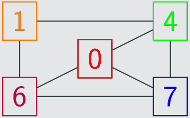

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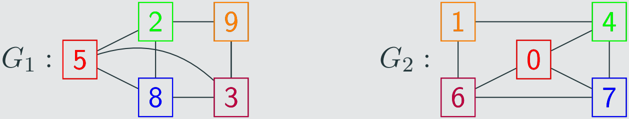
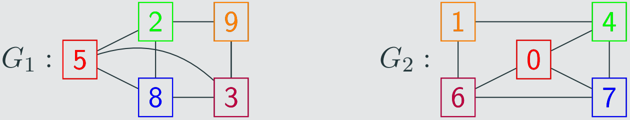
Graph isomorphism



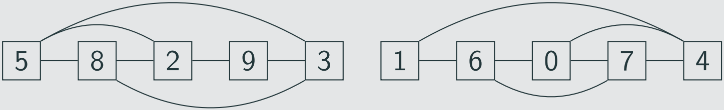
Reconfiguration

Overview

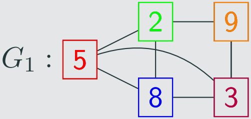
Graph isomorphism

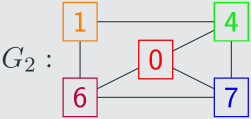


Reconfiguration



Graph isomorphism





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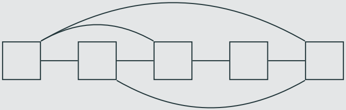

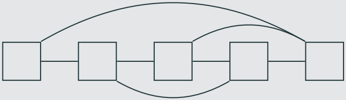

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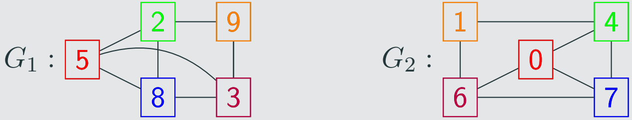

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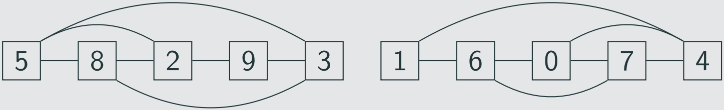

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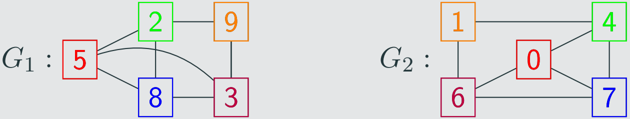

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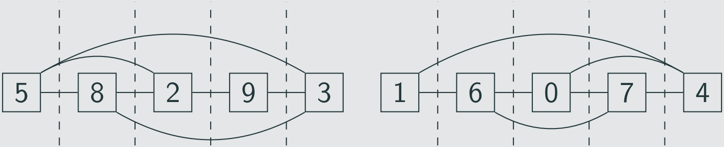

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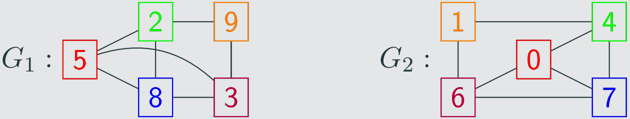

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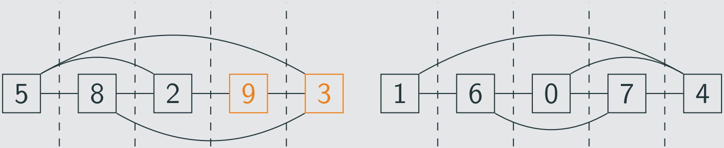

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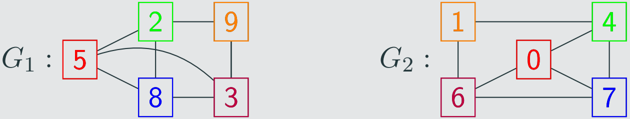

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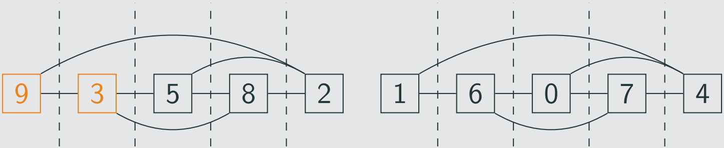
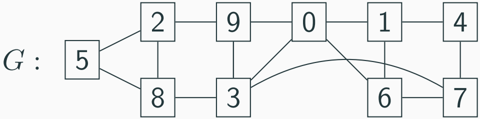


Order Reconfiguration

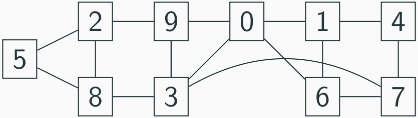
Cutwidth



Definition (Cutwidth of an ordering)

Cutwidth

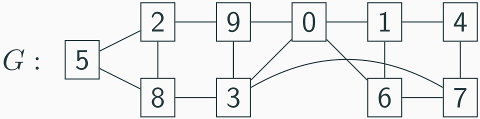
G :



Definition (Cutwidth of an ordering)

ω : 5 8 2 9 0 3 1 6 7 4

Cutwidth



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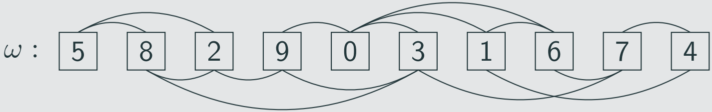
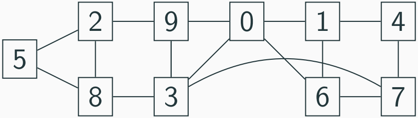


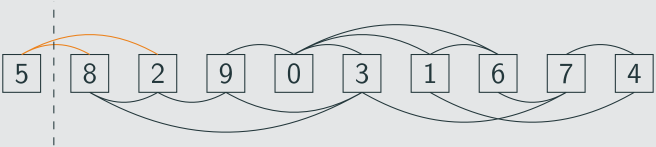
Cutwidth

G :



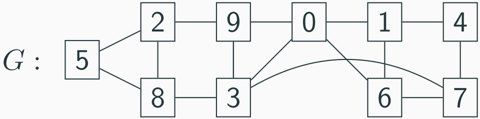
Definition (Cutwidth of an ordering)

ω :



$\mathrm{cw}(G, \omega) = \max\left(\{2,\right.$

Cutwidth



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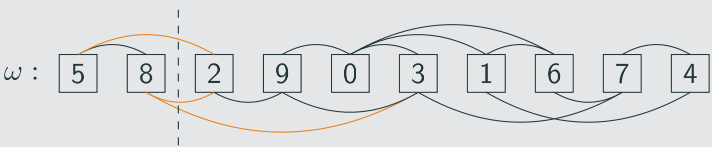
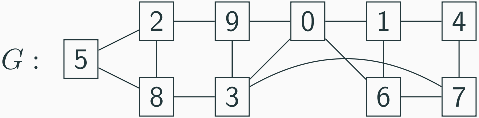


$\mathrm{cw}(G, \omega) = \max\left(\{2, 3,\right.$

Cutwidth



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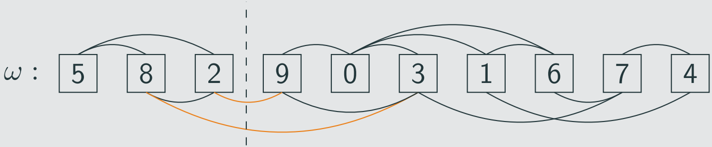
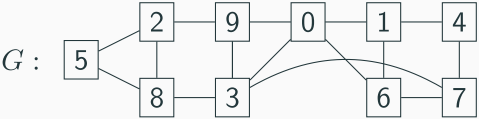


$\mathrm{cw}(G, \omega) = \max\left(\{2, 3, 2,\right.$

Cutwidth



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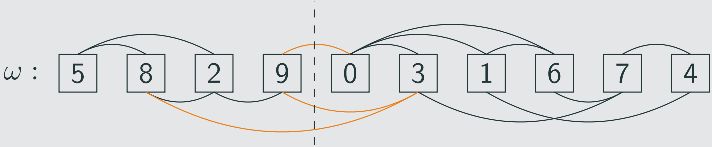
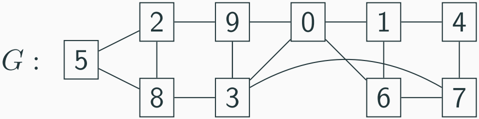


$\mathrm{cw}(G, \omega) = \max\left(\{2, 3, 2, 3,\right.$

Cutwidth



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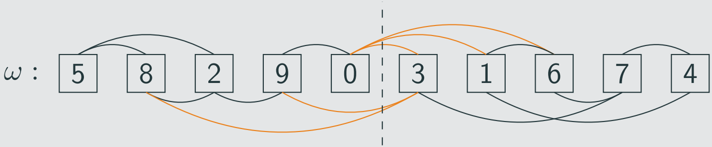
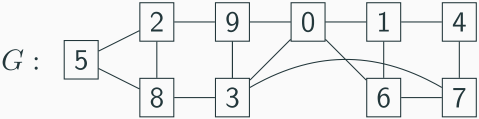


$\mathrm{cw}(G, \omega) = \max(\{2, 3, 2, 3, 5,$

Cutwidth



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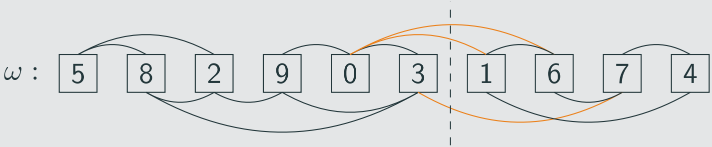
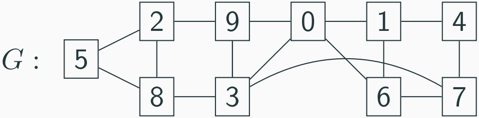


$\mathrm{cw}(G, \omega) = \max\left(\{2, 3, 2, 3, 5, 3,\right.$

Cutwidth



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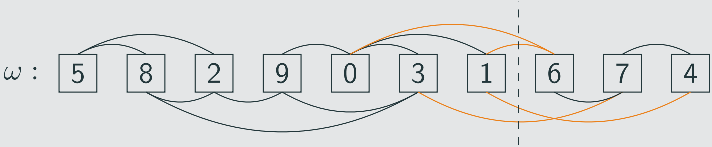
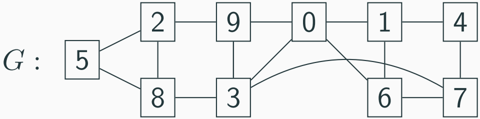


$\text{cw}(G, \omega) = \max\left(\{2, 3, 2, 3, 5, 3, 4,\right.$

Cutwidth



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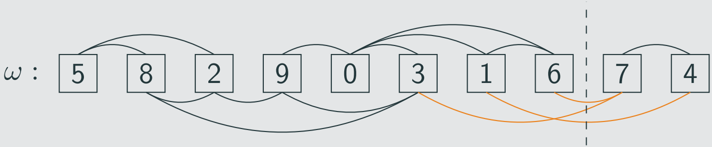
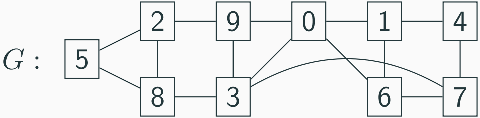


$\mathrm{cw}(G, \omega) = \max\left(\{2, 3, 2, 3, 5, 3, 4, 3,\right.$

Cutwidth



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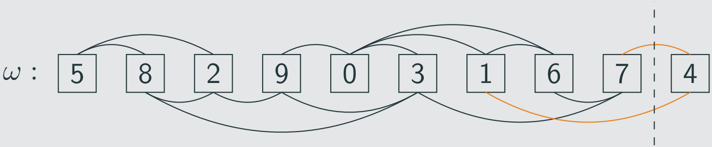
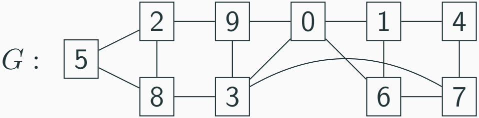
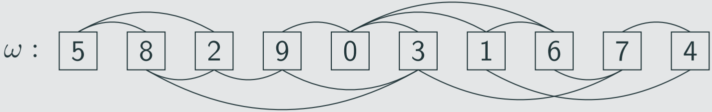


$\mathrm{cw}(G, \omega) = \max(\{2, 3, 2, 3, 5, 3, 4, 3, 2\}) = 5$

Cutwidth



Definition (Cutwidth of an ordering)



$\mathrm{cw}(G, \omega) = \max\left(\{2, 3, 2, 3, 5, 3, 4, 3, 2\}\right) = 5$

Definition (Cutwidth)

$\mathrm{cw}(G) = \min_{\omega}\left(\mathrm{cw}(G, \omega)\right)$

Order Reconfiguration

Definition (Swap)

5	8	2	9	0	3	1	6	7	4

Order Reconfiguration

Definition (Swap)

5 8 2 9 0 **3** **1** 6 7 4

Order Reconfiguration

Definition (Swap)

Order Reconfiguration

Definition (Swap)

5	8	2	9	0	1	3	6	7	4

Definition (Order reconfiguration)

ω can be reconfigured into ω' if

$$\omega = \omega_0 \to \omega_1 \to \cdots \to \omega_r = \omega'.$$

Order Reconfiguration

Definition (Swap)

5 8 2 9 0 1 3 6 7 4

Definition (Order reconfiguration)

ω can be reconfigured into ω' if

$$\omega = \omega_0 \to \omega_1 \to \cdots \to \omega_r = \omega'.$$

Problem (Bounded Cutwidth Order Reconfiguration)

Let G be an n-vertex graph, $\omega, \omega' : [n] \to V(G)$ be linear orders on the vertex set of G, and $k \in \mathbb{N}$. Is it true that ω can be reconfigured into ω' in cutwidth at most k?

Bounded Cutwidth Order Reconfiguration

Theorem

Let G be a graph and ω, ω' be linear orders of $V(G)$ of cutwidth at most k. Then, ω can be reconfigured into ω' in cutwidth at most $\mathrm{cw}(G, \omega) + \mathrm{cw}(G, \omega') \leq 2k$.

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

ω :	5	8	2	9	0	3	1	6	7	4

Proof: Big Steps

ω : 5 8 2 9 0 3 1 6 7 4

ω' : 0 1 2 3 4 5 6 7 8 9

ω : 5 8 2 9 0 3 1 6 7 4

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_1 \omega$:	0	5	8	2	9	3	1	6	7	4

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_1 \omega$:	0	5	8	2	9	3	1	6	7	4

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_1 \omega$:	0	5	8	2	9	3	1	6	7	4

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_2 \omega$:	0	1	5	8	2	9	3	6	7	4

Proof: Big Steps

ω : 5 8 2 9 0 3 1 6 7 4

ω' : 0 1 2 3 4 5 6 7 8 9

$\omega' \oplus_2 \omega$: 0 1 5 8 2 9 3 6 7 4

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_3 \omega$:	0	1	2	5	8	9	3	6	7	4

Proof: Big Steps

ω : 5 8 2 9 0 3 1 6 7 4

ω' : 0 1 2 3 4 5 6 7 8 9

$\omega' \oplus_3 \omega$: 0 1 2 5 8 9 3 6 7 4

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_4 \omega$:	0	1	2	3	5	8	9	6	7	4

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_5 \omega$:	0	1	2	3	4	5	8	9	6	7

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_6 \omega$:	0	1	2	3	4	5	8	9	6	7

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_7 \omega$:	0	1	2	3	4	5	6	8	9	7

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_8 \omega$:	0	1	2	3	4	5	6	7	8	9

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_9 \omega$:	0	1	2	3	4	5	6	7	8	9

Proof: Big Steps

ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

ω' :	0	1	2	3	4	5	6	7	8	9

Proof: Big Steps

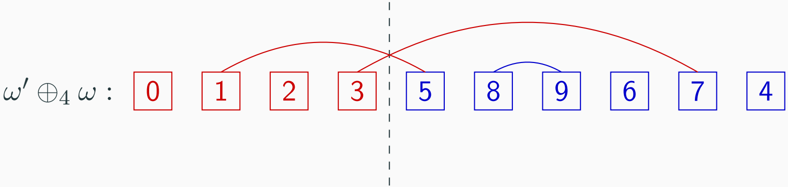
ω :	5	8	2	9	0	3	1	6	7	4
ω' :	0	1	2	3	4	5	6	7	8	9

$\omega' \oplus_4 \omega$:	0	1	2	3	5	8	9	6	7	4

Proof: Big Steps

ω : 5 8 2 9 0 3 1 6 7 4

ω' : 0 1 2 3 4 5 6 7 8 9



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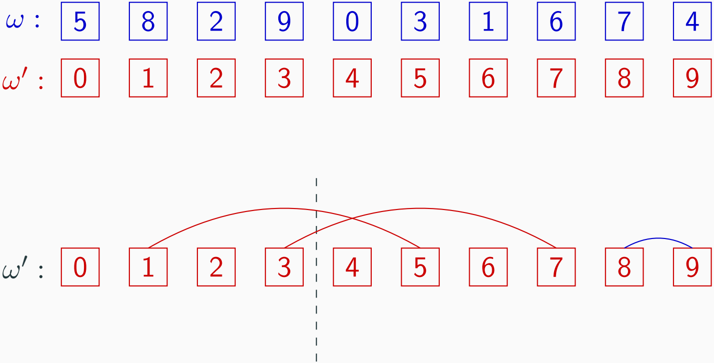

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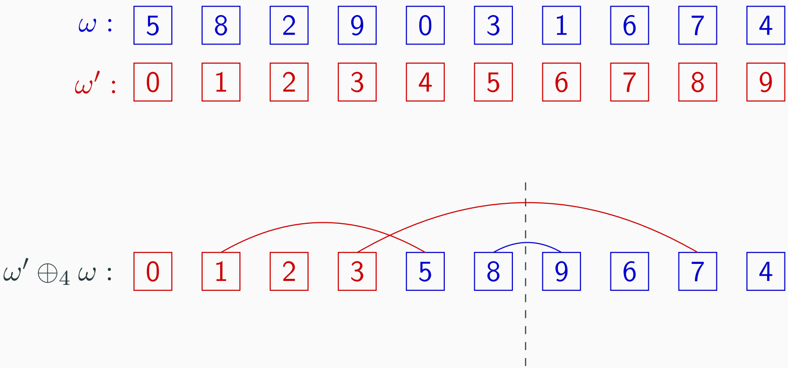

Proof: Small Steps

$\omega' \oplus_4 \omega$:	0	1	2	3	5	8	9	6	7	4
$\omega' \oplus_5 \omega$:	0	1	2	3	4	5	8	9	6	7

Proof: Small Steps

$\omega' \oplus_4 \omega$:	0	1	2	3	5	8	9	6	7	4
$\omega' \oplus_5 \omega$:	0	1	2	3	4	5	8	9	6	7

0	1	2	3	5	8	9	6	7	4

Proof: Small Steps

$\omega' \oplus_4 \omega$:	0	1	2	3	5	8	9	6	7	4
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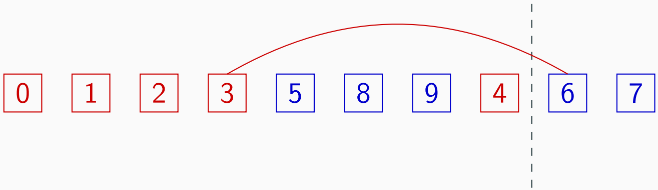

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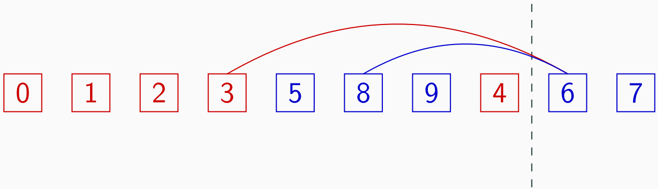

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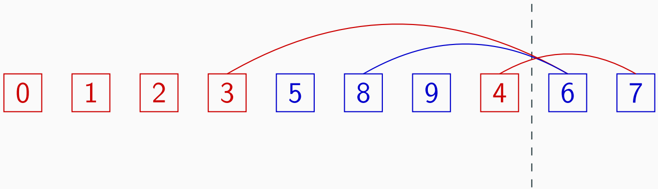
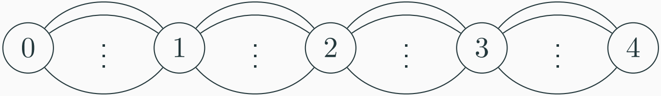
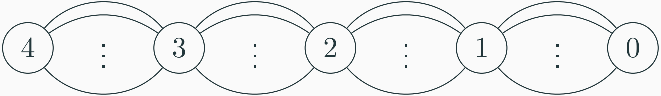


Lower Bound Construction





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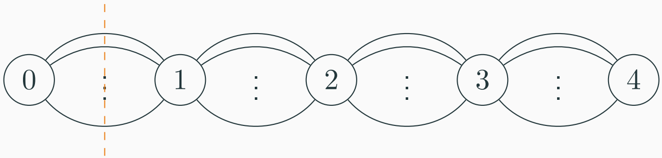

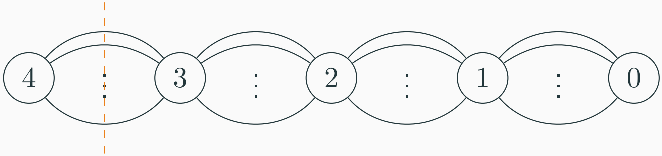

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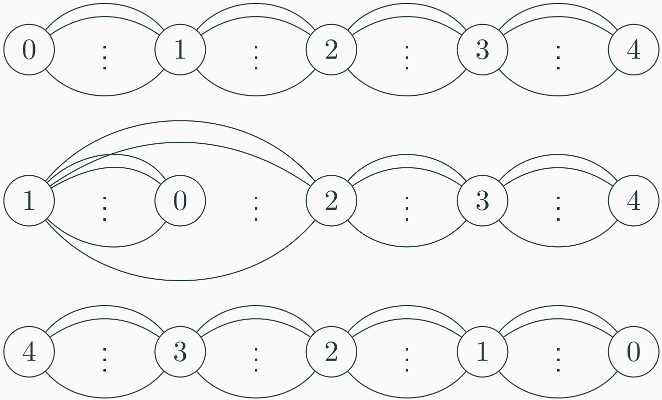

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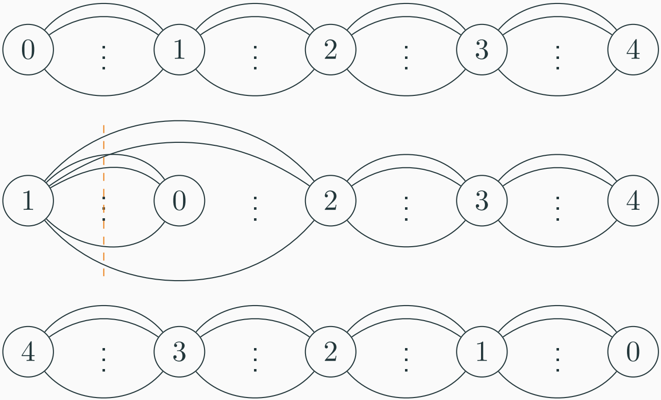
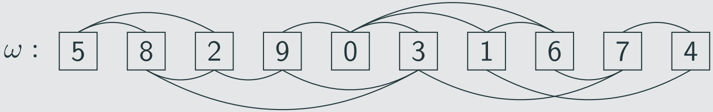


Bounded Vertex Separation Number Order Reconfiguration

Definition (Vertex separation number)



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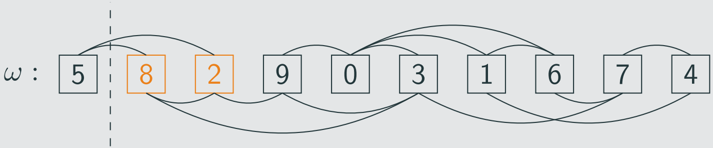

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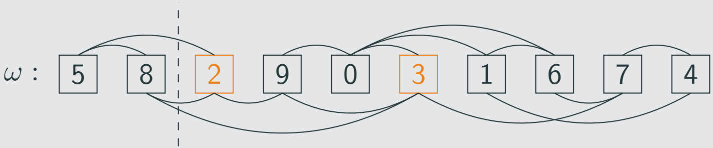

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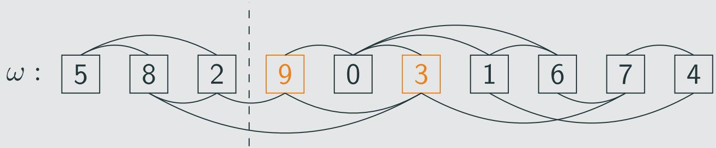

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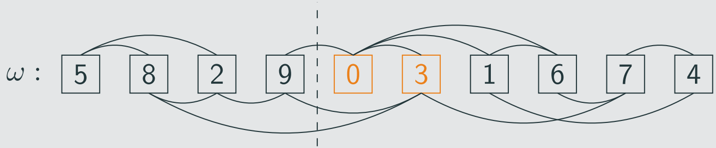

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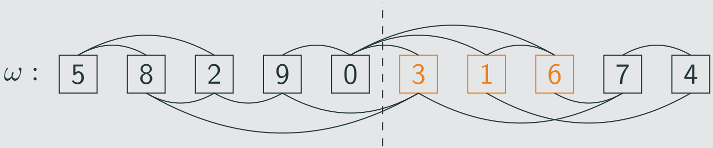

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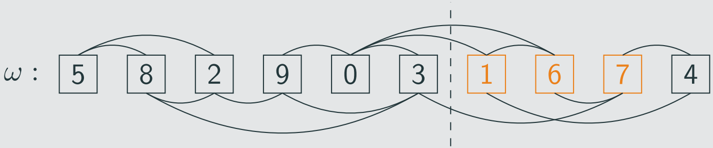

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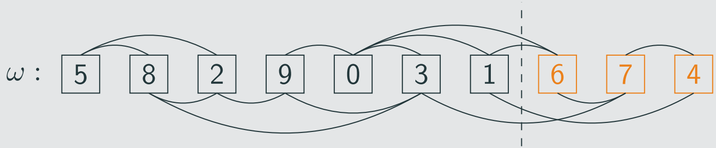

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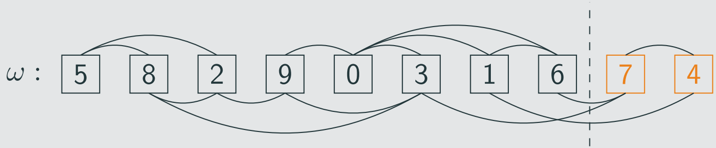

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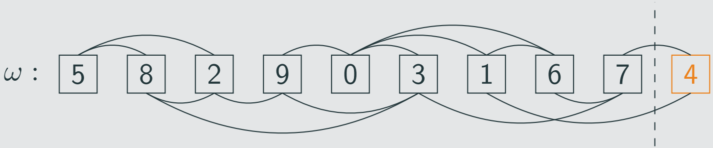

$\mathrm{vsn}(G, \omega) = \max(\{2, 2, 2, 2, 3, 3, 3, 2, 1\}) = 3$

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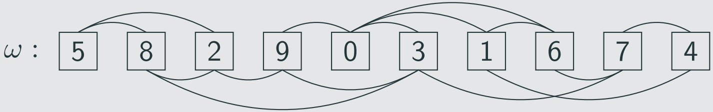

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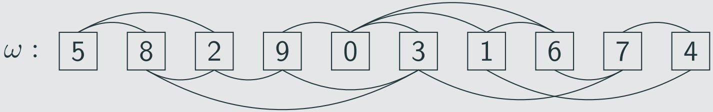

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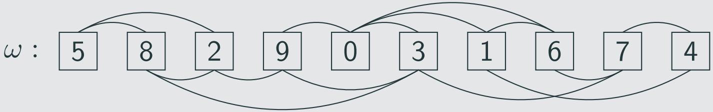

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Theorem

Let G be a graph and ω, ω' be linear orders of $V(G)$ of vertex separation number at most k. Then, ω can be reconfigured into ω' in vertex separation number at most $\mathrm{vsn}(G, \omega) + \mathrm{vsn}(G, \omega') \leq 2k$.

Slice rewriting system

String Rewriting System

Definition (String Rewriting System)

A string rewriting system is a pair (Σ, R) where Σ is a finite alphabet, and $R \subseteq \Sigma^* \times \Sigma^*$ is a set of rewriting rules.

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With the rule $ab \to cd$, we can rewrite $abba$ into $cdba$.

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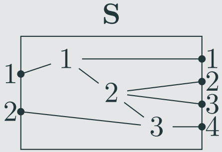
Problem (Reachability)

Given two strings w and w' in Σ^*, is there a sequence of rewrites that transforms w into w'?

Slices

Definition (Slice)

A slice is a (multi-)graph $G = (V, E)$ such that $V = I \cup C \cup O$.



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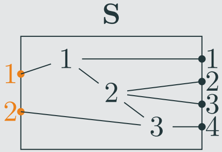

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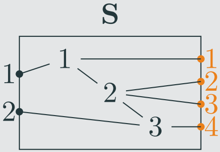

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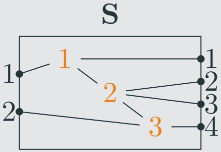

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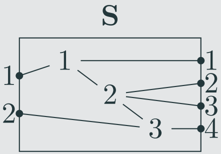

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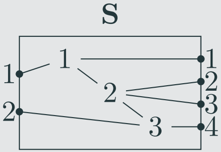

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Definition (Unit Slice)

Slices

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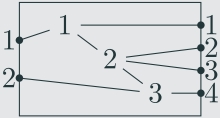
$\mathbf{S}$



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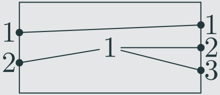
Definition (Unit Slice)

$\mathbf{S}_u$



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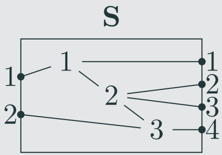

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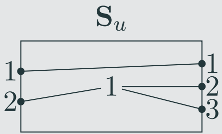

For each $k \in \mathbb{N}$, we define the alphabet $\mathbf{\Sigma}(k)$ as the set of all unit slices of width at most k.

Definition (Gluing)

Gluing of Slices

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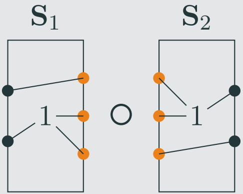

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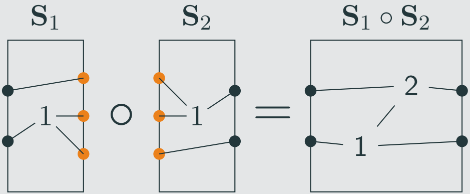

Unit Decompositions

Definition (Unit Decomposition)

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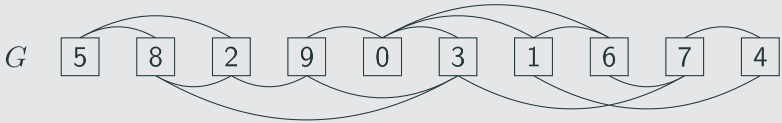

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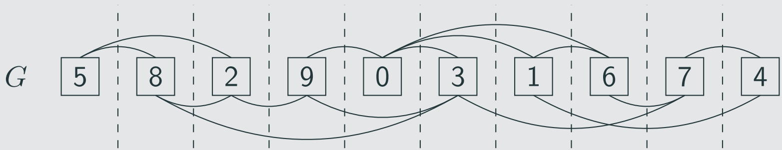

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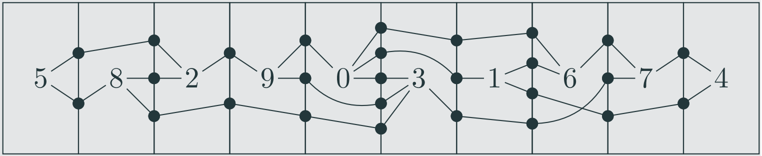
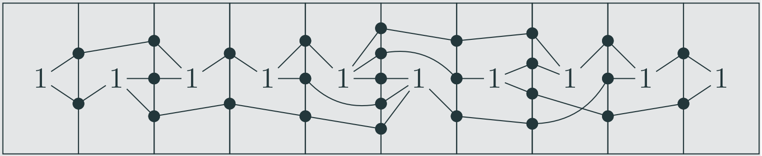


Definition (Unit Decomposition)

$\mathbf{U}_G$



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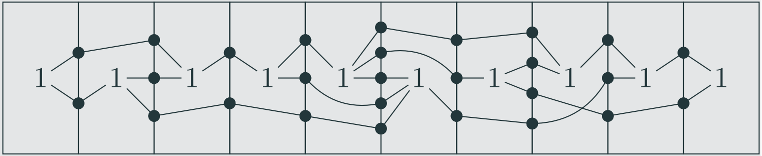
$\mathbf{U}_G$ 

- The gluing $\mathring{\mathbf{U}}_G$ of a unit decomposition $\mathbf{U}_G$ of a graph G is isomorphic to G.

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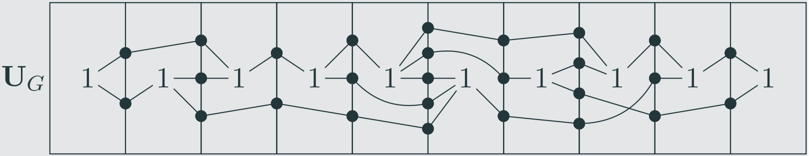

- The gluing $\mathring{\mathbf{U}}_G$ of a unit decomposition $\mathbf{U}_G$ of a graph G is isomorphic to G.
- $\mathbf{U}_G$ defines a linear order $\omega_{\mathbf{U}_G}$ of $V(\mathring{\mathbf{U}}_G)$.

Slice Equivalence Relation

Definition (Equivalence of unit slices)

$\mathbf{S}_1\mathbf{S}_2 \sim \mathbf{S}'_1\mathbf{S}'_2$ iff there exist an isomorphism φ from $\mathbf{S}_1 \circ \mathbf{S}_2$ to $\mathbf{S}'_1 \circ \mathbf{S}'_2$

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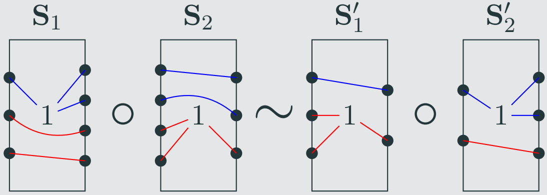

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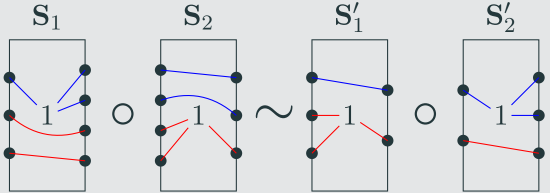
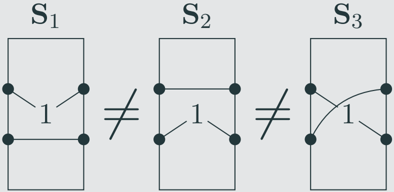


Definition (Slice rewriting system)

$\mathcal{R}(k) = \{\mathbf{S}_1\mathbf{S}_2 \to \mathbf{S}_1'\mathbf{S}_2' \ : \ \mathbf{S}_1\mathbf{S}_2 \sim \mathbf{S}_1'\mathbf{S}_2'\} \subseteq \mathbf{\Sigma}(k)^2 \times \mathbf{\Sigma}(k)^2$

Slice Equality and Twisting

Equality



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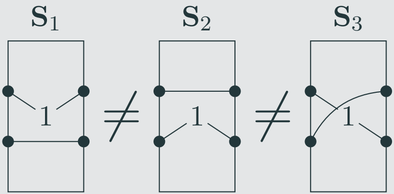

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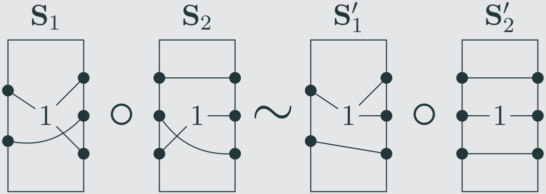

Graph Isomorphism and Reachability

Theorem

Let $\mathbf{U}$ *and* $\mathbf{U}'$ *be unit decompositions in* $\mathbf{\Sigma}(k)^{\circledast}$*. Then,* $\mathring{\mathbf{U}}$ *is isomorphic to* $\mathring{\mathbf{U}}'$ *if and only if* $\mathbf{U}'$ *is reachable from* $\mathbf{U}$ *using* $\mathcal{R}(2k)$*.*

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Theorem (Giannopoulou et al. Algorithmica 2019)

Let G be an n-vertex graph of cutwidth k. We can compute a linear order ω of the vertices of G of width k in time $k^{\mathcal{O}(k^2)} \cdot n$.

Theorem

Graph isomorphism for n-vertex graphs of cutwidth at most k can be reduced in time $k^{\mathcal{O}(k^2)} \cdot n$ to $\mathcal{R}(2k)$-reachability.

Reconfiguring Orders in General

The General Picture

- Reconfiguration problems where solutions are given by orderings are an interesting area.

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- Many concrete 'classical reconfiguration questions' are open / untouched, e.g.: What is the complexity of the next problem: Given two orderings ω, ω' of cutwidth $\leq k$ of a graph G and an integer $\ell \geq k$, is it possible to reconfigure ω into ω', so that all intermediate orderings have cutwidth $\leq \ell$?

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- These have also practical 'dynamic aspects', as explained with two examples next.

One Side Crossing Minimization (OSCM)

Definition (Two-layer drawing)

Let $G = (V_1, V_2, E)$ be a **bipartite** graph.

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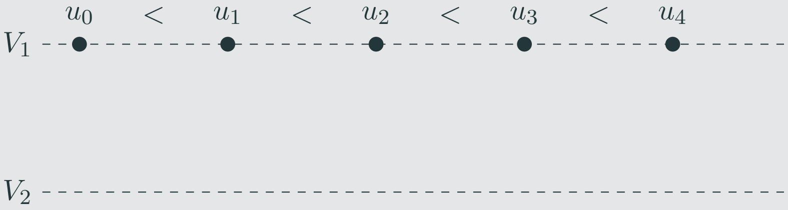

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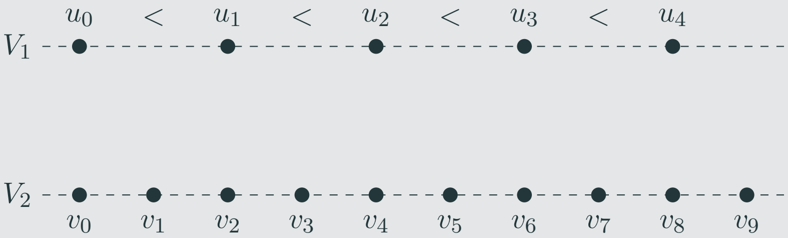

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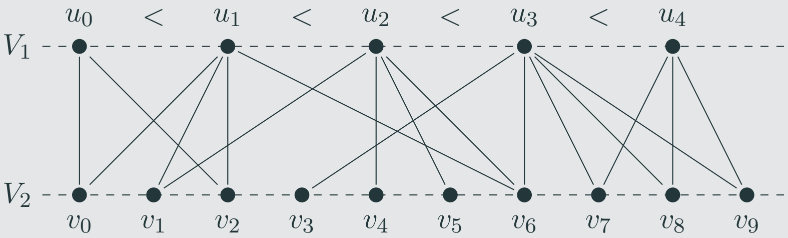

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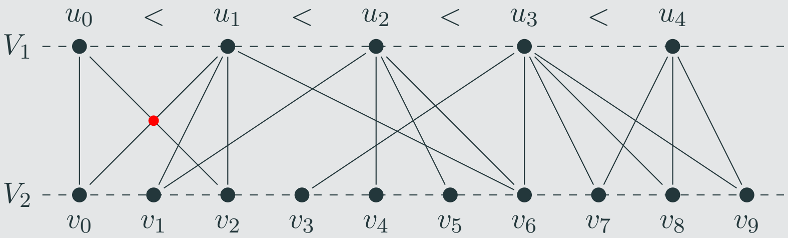

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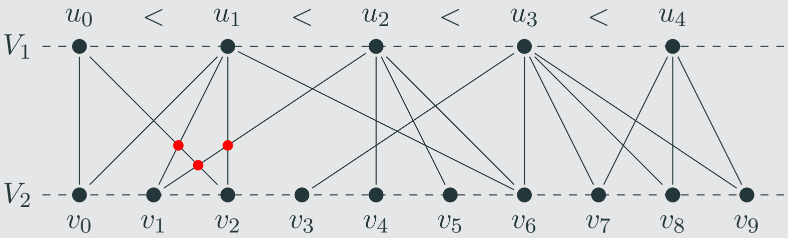

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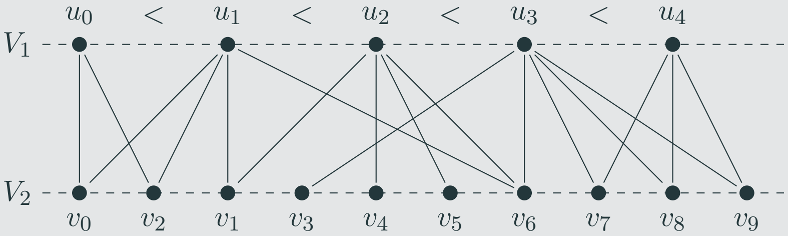

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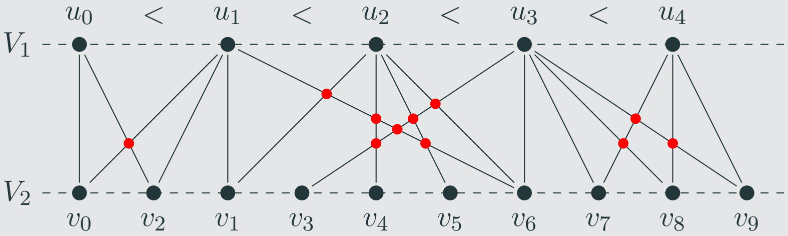

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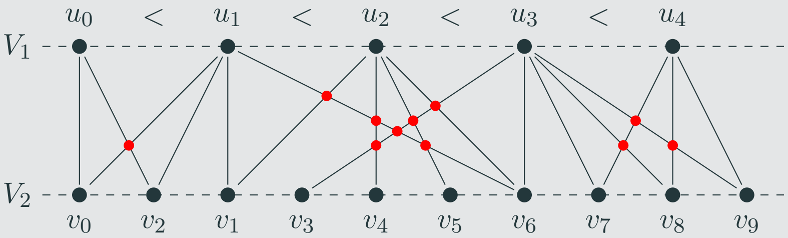

Problem (OSCM)

*Given a **bipartite graph** $G = (V_1, V_2, E)$, a **linear order** τ_1 on V_1 and $k \in \mathbb{N}$. Is there a **linear order** τ_2 on V_2 such that the two-layer drawing specified by (τ_1, τ_2) has at most k edge crossings?*

Grouping by Swapping (GbS)

Definition (Swap)

c	a	d	d	a	a	b	c	c	d	d

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c	a	d	d	a	a	b	c	c	d	d

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c a d d a b a c c d d

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Definition (Swap)

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Definition (Block string)

Grouping by Swapping (GbS)

Definition (Swap)

c a d d a b a c c d d

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a a a c c c b d d d d

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c a d d a b a c c d d

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a a a	c c c	b	d d d d

Grouping by Swapping (GbS)

Definition (Swap)

c a d d a b a c c d d

Definition (Block string)

| a a a | c c c | b | d d d d |

Problem (GbS)

*Given a **finite alphabet** Σ, a **string** $w \in \Sigma^*$, and $k \in \mathbb{N}$. Can we transform w in a **block string** w' with at most k **swaps**?*

Reduction from GbS to OSCM

GbS

An alphabet $\Sigma = \{a, b, c, d\}$ and $w = caddaabccdd$.

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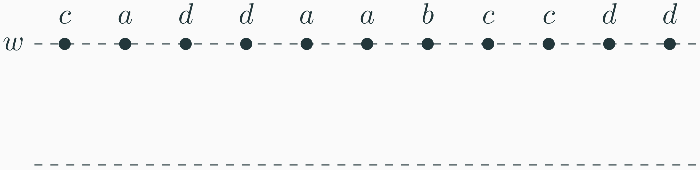

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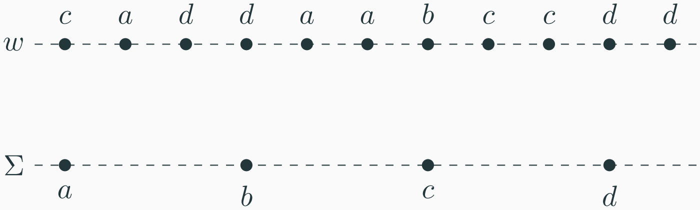

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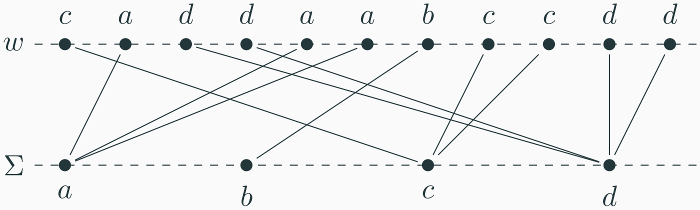

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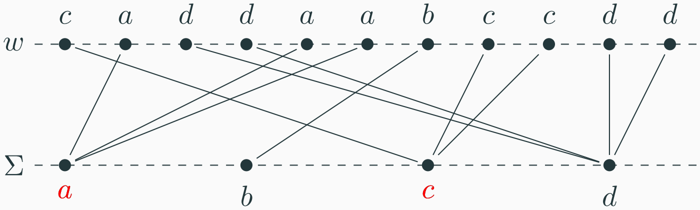

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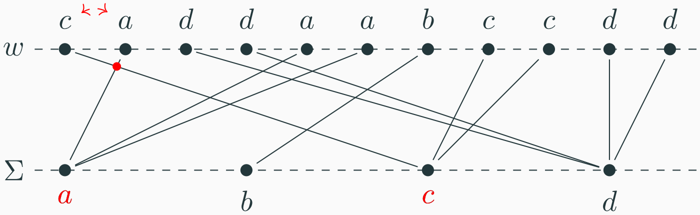

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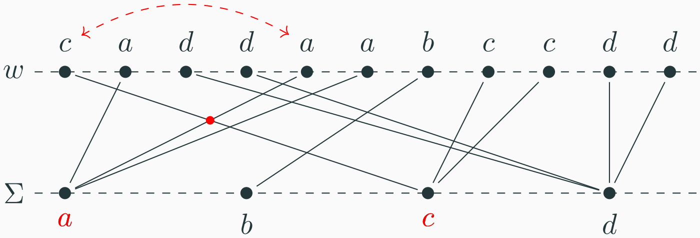

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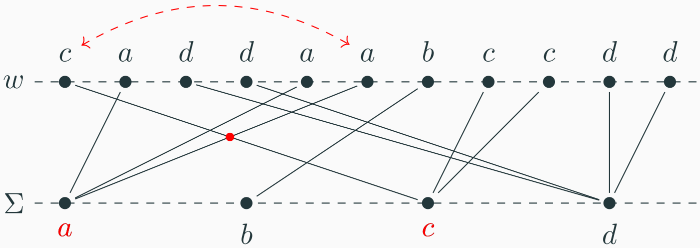

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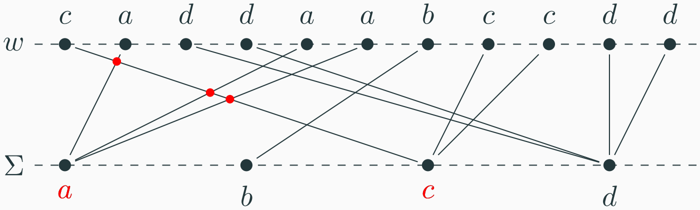

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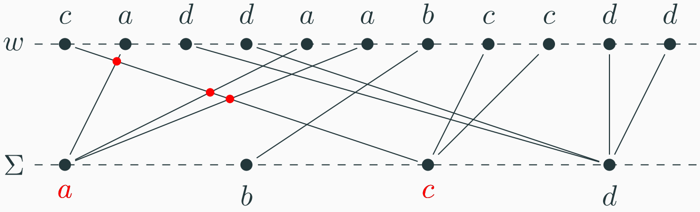

See E. Arrighi et al. FSTTCS 2020 & IJCAI 2021.

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- Conversely: we explained connections to string rewriting. Can we make use of other rewriting theory results in reconfiguration?

Thank you!

Trier, Germany
June 7th-9th

IWOCA 2022