Steenrod closed parameter ideals in $H^*(BA_4)$

Henrik Rüping

joint work with Marc Stephan and Ergün Yalçın

16.9.22

Motivating question

Let G be a finite group. Find the set of all tuples $(n_1, \ldots, n_k)$ such that there is a free G-action on $S^{n_1} \times \ldots \times S^{n_k}$.

Conjecture (Rank Conjecture for $G = (\mathbb{Z}/p)^m$)

For $G = (\mathbb{Z}/p)^m$ all tuples have $k \geq m$.

So this motivating question is way too hard.

Easier motivating question

Let $G = A_4$ be the alternating group on four elements. What is the set of all tuples $(n_1, \ldots, n_k)$ such that there is a free G-action on $S^{n_1} \times \ldots \times S^{n_k}$?

This motivating question is still too hard.

Oliver's result

Theorem (Oliver [Oli79])

There is no free A_4-action on $S^n \times \ldots \times S^n$ with $n \geq 1$.

Outline of the proof:

- If there is such an action on X, consider the map

$$H^*(BA_4; \mathbb{F}_2) \to H^*(X/A_4)$$

  induced by the classifying map. Oliver showed its kernel I is generated as an ideal by k elements of degree $n+1$;
- The quotient $H^*(BA_4)/I$ is finite, as it is in $H^*(X/A_4)$;
- The ideal is automatically closed under Steenrod operations;
- Oliver showed all Steenrod closed ideals ideals generated by elements of the same degree are generated by only a single element v^i.
- But $H^*(BA_4)/\langle v^i \rangle$ is infinite for $i > 0$.

Group cohomology

- $H^*(B(\mathbb{Z}/2)^2)$ is a polynomial ring $\mathbb{F}_2[a, b]$ on elements of degree one.
- $H^*(BA_4) = \mathbb{F}_2[a, b]^{C_3}$ where C_3 acts as $a \mapsto b \mapsto a + b$.
- A presentation is given as

$$H^*(BA_4) = \mathbb{F}_2[u, v, w]/(u^3 + v^2 + vw + w^2)$$

with

$$u = a^2 + ab + b^2$$
$$v = a^2b + ab^2$$
$$w = a^3 + a^2b + b^3$$

Especially u, v generate a polynomial ring in $H^*(BA_4)$.

Steenrod squares

The total Steenrod square Sq is given by the ring homomorphism sending $a \mapsto a + a^2$ and $b \mapsto b + b^2$. Thus we have:

$$\begin{aligned} \mathrm{Sq}(u) &= u + v + u^2 \\ \mathrm{Sq}(v) &= v + uv + v^2 \\ \mathrm{Sq}(w) &= w + u^2 + u(v + w) + w^2 \end{aligned}$$

Definition

An ideal I is Steenrod closed if $\mathrm{Sq}(I) \subset I$.

Example

$\langle u, v\rangle$ is Steenrod closed, $\langle u, v^2\rangle$ is not.

Restricting to a product of two spheres

Question (Blaszczyk)

What is the set of tuples (n_1, n_2) such that A_4 acts freely on $S^{n_1} \times S^{n_2}$?

This set is actually known to be nonempty. We can provide finer obstructions. However, even this question is much too hard for us.

Our results

Proposition

If A_4 acts freely on $X = S^{n_1} \times S^{n_2}$, then the kernel I of $H^*(BA_4) \to H^*(X/A_4)$ is generated by two elements in degrees $n_1 + 1$ and $n_2 + 1$.

Actually

Proposition

If A_4 acts freely on a finite CW complex X such that $H^*(X)$ is a four dimensional $\mathbb{F}_2$-vector space with basis $1, r, s, rs$, then the kernel I of $H^*(BA_4) \to H^*(X/A_4)$ is generated by two elements x, y in degrees $|r| + 1$ and $|s| + 1$.

Since $H^*(X/A_4)$ is still finite, the two elements x, y must form a system of parameters and thus they are coprime.

Definition

An ideal is a Steenrod closed parameter ideal, if it is Steenrod closed and a parameter ideal, i.e., generated by a system of (homogeneous) parameters $x, y \in H^*(BA_4)$.

Question

Can we classify all Steenrod closed parameter ideals in $H^*(BA_4)$?

Answer:Yes. That is what we really did. But it is complicated...

Especially if there is no such ideal with parameters of degrees $n_1 + 1, n_2 + 1$, then there cannot be a free action of A_4 on $S^{n_1} \times S^{n_2}$.

The twisted case

Recall that if x has degree n, then the degree $n+1$ component of $\mathrm{Sq}(x)$ is also called $\mathrm{Sq}^1(x)$.

Exercise

Show that Sq^1 is a derivation, e.g. $\mathrm{Sq}^1(xy) = \mathrm{Sq}^1(x)y + x\,\mathrm{Sq}^1(y)$ and thus $\mathrm{Sq}^1(x^2) = 0$.

Definition

A Steenrod closed parameter ideal is called *twisted*, if it is of the form $\langle x, \mathrm{Sq}^1(x)\rangle$.

Example

$\langle u, v\rangle$,$\langle u^3 + v^2, vu^2\rangle$ are twisted.

Theorem (R-Stephan-Yalçın)

The twisted Steenrod closed parameter ideals are all ideals of the form $\langle x_n, \mathrm{Sq}^1(x_n)\rangle$, where $x_1 = u$ and $x_{n+1} = ux_n^2 + \mathrm{Sq}^1(x_n)^2$.

The fibered case

Definition

A Steenrod closed parameter ideal is called *fibered*, if it has a system of parameters x, y such that $\langle x \rangle$ is Steenrod closed.

x need not be the generator of smaller degree. By Oliver's result, we have $x = v^k$ for some k.

Theorem (R-Stephan-Yalçın)

All fibered ideals are of the form $\langle v^k, u^l \rangle$ where k is not larger than the highest power of two dividing l.

Meyer and Smith show that an ideal of the form $\langle v^k, u^l \rangle$ is Steenrod closed, if and only if the condition from the theorem holds. The main work was to show that y can always be chosen as u^l.

Constructing new Steenrod closed parameter ideals

Exercise

If $\langle x, y\rangle$ is a Steenrod closed parameter ideal, so is $\langle x^2, y^2\rangle$.

Proof.

Use that squaring is a ring homomorphism,e.g. if $\mathrm{Sq}(x) = \alpha x + \beta y$, we then have $\mathrm{Sq}(x^2) = \alpha^2 x^2 + \beta^2 y^2$. □

Exercise

If $\langle x, y\rangle$ and $\langle v^n x, y\rangle$ are Steenrod closed parameter ideals, so is $\langle v^i x, y\rangle$ for all $1 \leq i \leq n$.

Using the exercises to find the remaining ideals

Let x_n be the generator of a twisted ideal. Then the following ideals are Steenrod closed:

- $\langle x_n^2, \mathrm{Sq}^1(x_n)^2\rangle = \langle x_n^2, ux_n^2 + \mathrm{Sq}^1(x_n)^2\rangle$
- $\langle x_{n+1}, \mathrm{Sq}^1(x_{n+1})\rangle = \langle vx_n^2, ux_n^2 + \mathrm{Sq}^1(x_n)^2\rangle$

After raising the generators to the 2^m-th power for some m, we can use the second exercise to show that
$\langle v^i x_n^{2^{m+1}}, u^{2^m} x_n^{2^{m+1}} + \mathrm{Sq}^1(x_n)^{2^{m+1}}\rangle = \langle v^i x_n^{2^{m+1}}, x_{n+1}^{2^m}\rangle$ for $1 \leq i \leq 2^m$ is also a Steenrod closed parameter ideal.

Definition

We call these ideals *mixed*.

The Main Theorem

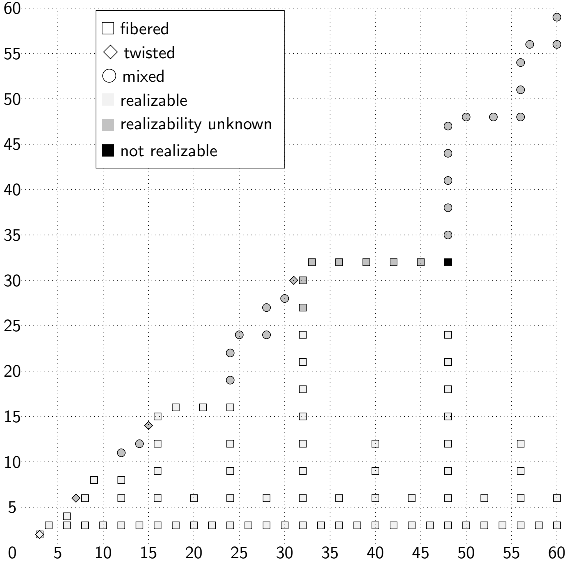
Theorem (R-Stephan-Yalçın)

Any Steenrod closed parameter ideal is fibered, twisted or mixed.

Theorem (R-Stephan-Yalçın)

For any given pair of natural numbers there is at most one Steenrod closed parameter ideal whose generators are in these dimensions.

Steenrod closed parameter ideals up to degree 60



Realizability

Question

What are the known constructions for free A_4-actions on products of spheres.

More precise:

Question

Given a Steenrod closed parameter ideal. Is there a free action on a nice space X such that the given ideal is the kernel of $H^*(BA_4) \to H^*(X/A_4)$?

Realizing fibered ideals using fixity methods

Theorem (R-Stephan-Yalçın)

Any fibered ideal $I = \langle v^k, u^l \rangle$ with $k \leq 8$ can be realized by a free A_4-action on the total space of an S^{2l-1}-bundle over S^{3k-1}. If $I \neq \langle v^c, u^c \rangle$ for all $c = 1, 2, 4, 8$, then I can be realized by a trivial bundle, and thus by a free A_4-action on a product of spheres.

This was already known for $k \leq 4$. We then extended the constructions to the octonionic case. In the four excluded ideals one cannot find another action on a product of spheres.

Realizing fibered ideals using Adem-Smith methods

Theorem (R-Stephan-Yalçın)

For every $k \geq 1$, there exists an integer $l_0 \geq 1$, depending on k, such that for every $s \geq 1$ the ideal $\langle v^k, u^{l_0 s} \rangle$ in $H^*(BA_4; \mathbb{F}_2)$ is realized by a free A_4-action on a finite CW-complex homotopy equivalent to a product of two spheres.

Nonrealizability

However there is the following strong obstruction:

Theorem (Meyer,Smith [MS03, Theorem 1.2])

$H^*(B(\mathbb{Z}/2)^2)/\langle u^{2^t}, v^{2^t}\rangle$ occurs as a cohomology algebra of a topological space if and only if $t = 0, 1, 2, 3$.

If there was a free action of A_4 on X as before, then the cohomology of $X/((\mathbb{Z}/2)^2)$ would be isomorphic to $H^*(B(\mathbb{Z}/2)^2)/\langle u^{2^t}, v^{2^t}\rangle$ and this contradicts the result of Meyer and Smith above.

Outlook

Question

What about the realizability of fibered ideals $\langle v^k, u^l \rangle$ for $k \geq 9$.

Question

What about other groups and primes?

Question

Is it possible to use higher cohomology operations to construct further obstructions to the realizability of Steenrod closed parameter ideals?

Question

What changes if we consider actions where C_3 is allowed to have fixed points?

Outlook: Mixed ideals

Question

Can the nonfibered ideals be realized by an action on a space?

The A_4-spaces realizing the fibered ideals so far always had the property that the projection to one of the factors is A_4-equivariant with respect to some non-free action on that factor. If one could realize nonfibered ideals, this would not work and we would need to construct an action mixing both coordinates.

Example

Is there a free A_4-action on $X = S^{11} \times S^{10}$ realizing $\langle u^6 + v^4, u^4 v \rangle$?

Details, proofs and a lot of computations can be found in [RSY22].
Thank you for your attention!

References I

Dagmar M. Meyer and Larry Smith, *Realization and nonrealization of Poincaré duality quotients of* $\mathbb{F}_2[x, y]$ *as topological spaces*, Fund. Math. **177** (2003), no. 3, 241–250. MR 1992242

Robert Oliver, *Free compact group actions on products of spheres*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Lecture Notes in Math., vol. 763, Springer, Berlin, 1979, pp. 539–548. MR 561237

Henrik Rüping, Marc Stephan, and Ergun Yalcin, *Steenrod closed parameter ideals in the mod-2 cohomology of* A_4 *and* $SO(3)$, 2022, `https://arxiv.org/abs/2206.11802`.