The Role of the Energy Scalar Product in the QNM Spectral Instability Problem

Edgar Gasperín

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work in collaboration with José Luis Jaramillo

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INSTITUT DE MATHÉMATIQUES
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Motivation

QNMs in a nutshell

- Eigenmodes of a non-self adjoint operator. $Lu = \lambda u$
- Dissipative systems: Optics, Seismology, ..., **Linearised GR. BH pert**.

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V(x) \right) \phi(t,x) = 0,$$

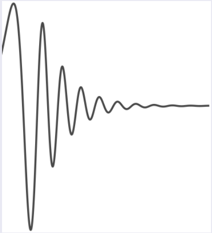
$$\phi_\omega(t,x) = e^{i\omega t}\phi(\omega, x),$$

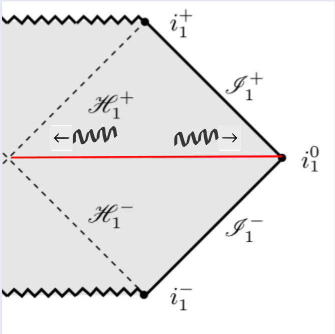
$$\left(\omega^2 + \frac{d^2}{dx^2} - V(x) \right) \phi(\omega,x) = 0,$$

outbc : $x \to \pm\infty \quad \phi(\omega,x) \sim e^{\mp i\omega x}$

Non Self-Adj $\omega = \omega^R + i\omega^I, \quad \omega^I > 0$

$x \to \pm\infty \quad \phi(\omega,x) \sim e^{\mp i\omega^R x} e^{\pm \omega^I x}$





Motivation

QNM GR: Cauchy slice vs Hyp slice

- QN modes are divergent at i^0
- $\rightsquigarrow$ "problem at i^0" $\rightsquigarrow$ (Pen, Fri)
- Sol: don't go to i^0! Go to $\mathscr{I}^+$

QNMs Instability

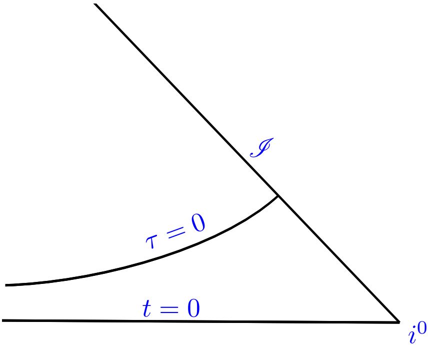
- Nöllert & Price: evidence of instability in Schw QNM spectrum (Cauchy)
- Jaramillo, Panosso-Macedo, Al-Sheikh: Introduction of pseudospectrum in GR (Hyp)

Main Ingredients

- Hyperboloidal slices
- Pseudospectrum

Non-normal & Non-self adjoint operators

- Adjoint $\longleftrightarrow$ Scalar Product
- Pseudospectrum $\longleftrightarrow$ Norm



$$t = \tau - h(x), \quad r_\star = g(x)$$

The ABC operator

Let Q defined on $x \in [x_0, x_1]$ be ($a \neq 0$)

$$Q = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c,$$

Formal Adjoint : $Q^\dagger$

$$\langle \phi_T, Q\phi \rangle = \langle Q^\dagger \phi_T, \phi \rangle$$

Q is Formally Normal if :

$$[Q, Q^\dagger] = QQ^\dagger - Q^\dagger Q = 0$$

L^2-inner product

$$\langle \phi_T, \phi \rangle_I = \int_{x_0}^{x_1} \bar{\phi}_T \phi dx$$

$$Q_I^\dagger = a\frac{d^2}{dx^2} - b\frac{d}{dx} + c$$

Q is not self adjoint respect to I.

← An illustrative example

Domains

$\mathcal{D} = \{\phi \in \mathcal{H} \mid \phi(x_0) = \phi(x_1) = 0\}.$

$D(Q) = D(Q^\dagger) = \mathcal{D}.$

$D(QQ^\dagger) \neq D(Q^\dagger Q)$ **formally** normal.

Sturm-Lioville inner product

Let $w = a^{-1}\exp(a^{-1}bx)$ $\quad p = aw$ and $q = -cw$.

$$Q = w^{-1}\left(\frac{d}{dx}\left(p\frac{d}{dx}\right) - q\right)$$

$$\langle \phi_T, \phi \rangle_w = \int_{x_0}^{x_1} \bar{\phi}_T \phi w dx$$

$Q_w^\dagger = Q$. Self adjoint!

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Pseudospectrum & the impact of the choice of norm

ϵ-pseudospectrum of an operator A

(Resolvent) $R_A(\lambda) \equiv (A - \lambda\mathbb{I})^{-1}$

(Spectrum) $\sigma(A) \equiv \{\lambda \in \mathbb{C} : \nexists R_A(\lambda)\}$

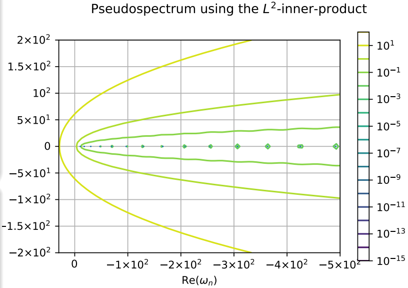
$\sigma^{\epsilon}(A) \equiv \{\lambda \in \mathbb{C} : ||R_A(\lambda)|| > 1/\epsilon\}$

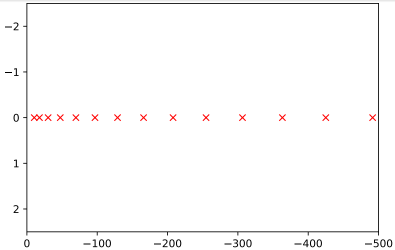
$\sigma^{\epsilon}(A) \equiv \{\lambda \in \sigma(A + \delta A), \ ||\delta A|| < \epsilon\}$

ABC operator example

$$\Phi = -(x - x_0)(x - x_1)\phi, \qquad Q \to \hat{Q}$$

$$\hat{Q} = A(x)\frac{d^2}{dx^2} + B(x)\frac{d}{dx} + C(x),$$





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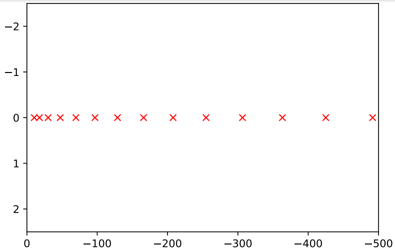
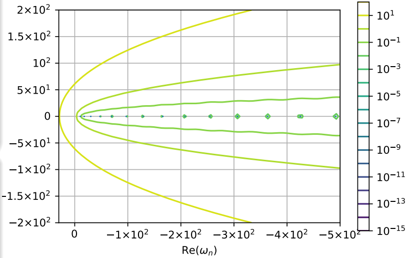
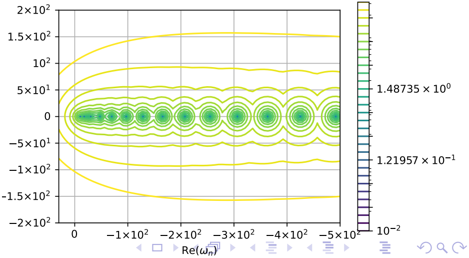


Pseudospectrum using the L^2-inner-product



Pseudospectrum using Gram Matrix = SturmLiouville-w



QNMs Hyp

Set up

$$\boxed{\Box_g \Phi = 0}$$

$$g \to dS^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\mathbb{S}^2,$$

$$\Phi = \frac{1}{r}\sum_{l,m} \phi_{\ell m} Y_{\ell m} \quad \Longrightarrow \quad \Box_{\mathring{\eta}}\phi - V(r,\ell)\phi = 0$$

$$\mathring{\eta} \to dS^2 = -dt^2 + dr^2$$

QNMHyp (Jaramillo, Panosso-Macedo, Al-Sheikh)

$$Lu = \lambda u, \qquad u = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \qquad L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right),$$

$$L_1 = w^{-1}(x)(\partial_x(p(x)\partial_x) - q(x))$$

$$L_2 = w^{-1}(x)(\gamma(x)\partial_x + \partial_x(\gamma(x)\,\cdot\,))$$

$$\mathring{E}(\phi) = \frac{1}{2}\int_{x_a}^{x_b} w\partial_\tau\phi\partial_\tau\bar{\phi} + p\partial_x\phi\partial_x\bar{\phi} + q\phi\bar{\phi}\,dx$$

Hyperboloidal coords

$$dr/dr_\star = f(r)$$

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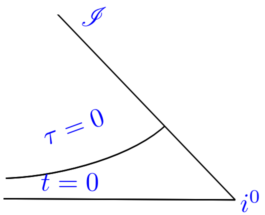
$$w = \frac{g'^2 - h'^2}{g'}, \gamma = \frac{h'}{g'}, p = \frac{1}{g'}$$



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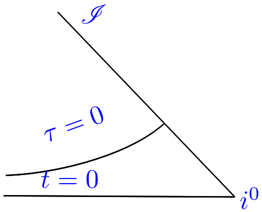

Physical energy vs effective energy (G. & Jaramillo 21)

Effective Energy $\mathring{E}(\phi)$

$\Box_{\mathring{\eta}}\phi - V\phi = 0.$ $\quad \mathring{\eta} =$ 2dim Mink, $\quad V = (r, \ell)$

$$\mathring{T}_{ab} = \frac{1}{2}\left(\mathring{\nabla}_a\bar{\phi}\mathring{\nabla}_b\phi - \frac{1}{2}\mathring{\eta}_{ab}\mathring{\eta}^{cd}\mathring{\nabla}_c\bar{\phi}\mathring{\nabla}_d\phi + V\phi\bar{\phi} + c.c.\right)$$

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$\Box_g\Phi = 0 \quad g$ – 4-dim, Spherically Sym

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Relation and Boundary Term

$$E = \sum_{lm}\mathring{E}_{lm} - \left(\frac{f}{2r}\phi_{lm}\bar{\phi}_{lm}\right)\Big|_{x_a}^{x_b}$$

Total energy flux

$$\partial_\tau E = F\Big|_{x_a}^{x_b},$$

where

$$F = \sum_{lm}\gamma\partial_\tau\phi_{lm}\partial_\tau\bar{\phi}_{lm} + p\,\mathrm{Re}(\partial_\tau\phi_{lm}\partial_x\bar{\phi}_{lm}) - \frac{f}{2r}\partial_\tau(\phi_{lm}\bar{\phi}_{lm}).$$

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In a nutshell

$$Lu = \lambda u \quad \text{test function } u_T = \begin{pmatrix} \phi_T \\ \psi_T \end{pmatrix} \implies \langle u_T, Lu \rangle_E = \lambda \langle u_T, u \rangle_E .$$

lower the number of derivatives using integration by parts

Weak Formulation QNMs

$$\int_{x_a}^{x_b} W_L(u_T, u)\, dx + W_B(u_T, u)\Big|_{x_a}^{x_b} = i\lambda \int_{x_a}^{x_b} W_R(u_T, u)\, dx,$$

$$W_L = p(\partial_x \bar{\phi}_T \partial_x \psi - \partial_x \phi \partial_x \bar{\psi}_T) + q(\bar{\phi}_T \psi - \bar{\psi}_T \phi) - w\psi L_2 \bar{\psi}_T,$$

$$W_R = p\partial_x \bar{\phi}_T \partial_x \phi + w\bar{\psi}_T \psi + q\bar{\phi}_T \phi$$

$$W_B = p\bar{\psi}_T \partial_x \phi + 2\gamma \bar{\psi}_T \psi$$

Proof of concept: Pöschl-Teller

$$V(r_\star) = V_o \operatorname{sech}^2(r_\star), r_\star \in (-\infty, \infty).$$

$$w = 1, p = 1 - x^2, q = V_o, \gamma = -x,$$

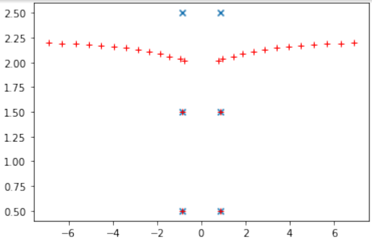

Figure: Pöschl-Teller QNM as a weak problem solved in FEniCS. N=500. Lagrange finite elements function space "CG" (Continuous Garlekin). In blue: exact eigs. In red: Num eigs

Resonant expansions (Jaramillo & Al-Sheikh, G. & Jar. 21)

Keldysh's expansion theorem

$$Lv_n = \lambda_n v_n, \quad L^\dagger w_n = \overline{\lambda}_n w_n$$

$$G_\lambda(x', x) \sim \sum_{\lambda_j \in \Omega} \frac{w_j^\dagger(x') v_j(x)}{\lambda - \lambda_j}$$

Keldysh + conditioning num + energy norm

$$\kappa_n = \frac{||w_n||\ ||v_n||}{\langle w_n, v_n \rangle}, \ \hat{w}_n = \frac{w_n}{||w_n||}, \ \hat{v}_n = \frac{v_n}{||v_n||}$$

$$R_L(\lambda) \sim \sum_{\lambda_n \in \Omega} \kappa_n \frac{|\hat{v}_n\rangle\langle \hat{w}_n|}{\lambda_n - \lambda}$$

Self adjoint case : $\kappa_n = 1$ & Conv in $\mathbb{C} \setminus \sigma(L)$

QNMs resonant expansion in hyp

$$\partial_\tau u = iLu \ ,$$
$$u(\tau = 0, x) = u_0(x) \quad , \quad ||u_0||_E < \infty$$

$$(L + is)u(s; x) = iu_0(x)$$

(to Fourier ω) $s = i\omega$

$$u(\tau, x) = \sum_{\omega_j \in \Omega} e^{i\omega_j \tau} \kappa_n \langle \hat{w}_j | u_0 \rangle_E \hat{v}_j + E_\Omega(\tau; u_0)$$

Bound on error

$$||E_\Omega(\tau; u_0)||_E \leq C_\Omega(a, L) e^{-a\tau} ||u_0||_E$$
$$a = \max\{\mathrm{Im}(\omega), \omega \in \Omega\}$$

Original field

$$\phi(\tau, x) \sim \sum_n e^{i\omega_j \tau} a_n \hat{\phi}_j^R$$

$$a_j = \frac{\kappa_j}{2} \Big(\langle \hat{\phi}_j^L, \varphi_0 \rangle_{H^1_{(V,p)}} - i\omega_j \langle \hat{\phi}_j^L, \varphi_1 \rangle_{(2,w)} \Big).$$

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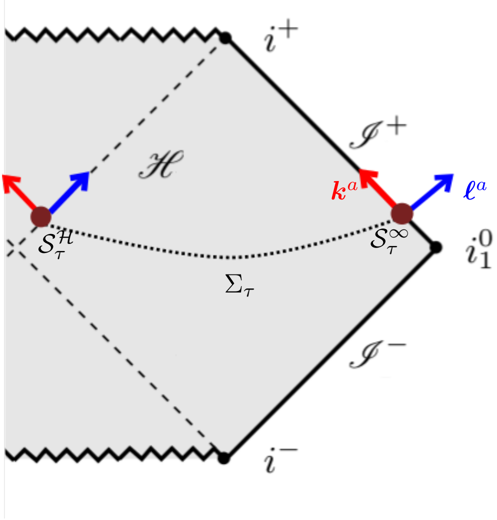
Perspectives & future work



BMS, GeoQNM, Normal mds (???)

$$F(\tau) = \int_{\mathcal{S}_\tau^{\mathcal{H}}} \gamma |\partial_\tau \phi|^2 dS + \int_{\mathcal{S}_\tau^{\infty}} \gamma |\partial_\tau \phi|^2 dS$$

$$L^\dagger = L + L^\partial \quad , \quad L^\partial = \frac{1}{i}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & L_2^\partial \end{array}\right)$$

$$L_2^\partial = -2\frac{(\gamma k^a)\ell_a}{w}\left(\delta_{\mathcal{S}_\tau^{\mathcal{H}}} - \delta_{\mathcal{S}_\tau^{\infty}}\right)$$

$$\xi^a = \gamma k^a \quad , \quad \mathcal{L}_k \gamma = 0$$

Thanks for listening