Toppleable permutations, acyclic orientations and excedances

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(joint with D. Hathcock and P. Tetali, `arXiv:2010.11236`, and with B. Bényi, `arXiv:2104.13654`)

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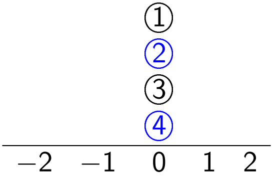
A sorting algorithm

- Defined by Hopkins, McConville and Propp (*EJC*, 2017).
- Start with chips labelled $1, \dots, n$ initially at the origin in $\mathbb{Z}$.
- At each time step, do the following:
  1. If no position has two or more chips, stop. Else, go to step 2.
  2. Choose a position i uniformly at random among positions occupied by more than one chip.
  3. Pick two chips uniformly from those at site i.
  4. If the two chips are α, β with $\alpha < \beta$, then move α to position $i-1$ and β to $i+1$.
  5. Go to step 1.

Example: $n = 4$



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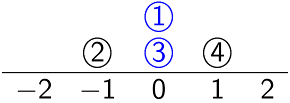

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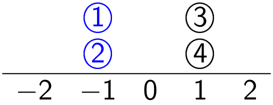

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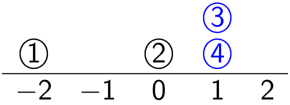

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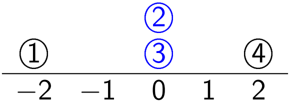

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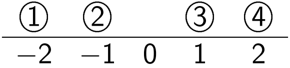


The main result

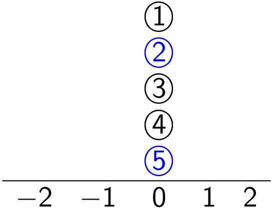
Theorem (Hopkins, McConville and Propp, *Elec. J. Comb.*, 2017)

When n is even, the chips end up at positions

$$-\frac{n}{2}, \ldots, -1, 1, \ldots, \frac{n}{2}$$

and are always sorted.

Example: $n = 5$



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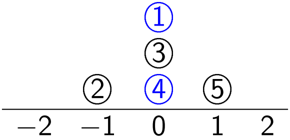

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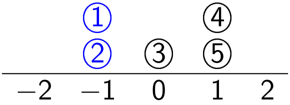

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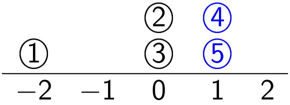

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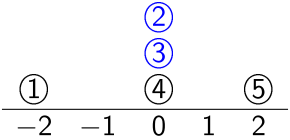


Example: $n = 5$

①	②	④	③	⑤
-2	-1	0	1	2

Open problem

When n is odd, one can show that the chips end up at positions

$$-\frac{n-1}{2}, \dots, \frac{n-1}{2}.$$

Conjecture (Hopkins, McConville and Propp, *Elec. J. Comb.*, 2017)

When n is odd, the chips get sorted with probability tending to $1/3$ as $n \to \infty$.

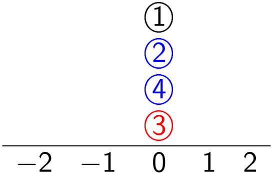
Further work

- *Root system chip firing*:
  1. Galashin, Hopkins, McConville and Postnikov (*SLC* 2018),
  2. Galashin, Hopkins, McConville and Postnikov (*Math. Z.* 2019),
  3. Hopkins and Postnikov (*Alg. Comb.* 2019).
- Progress towards proving the conjecture:
  1. Klivans and Liscio (*SLC* 2020),
  2. Felzenszwalb and Klivans (*JCTA* 2021).
  3. Klivans and Liscio (`arXiv:2006.12324`).

Modification of the process

- Suppose n is even and fix $r \in [n]$.
- Assume that the chip labelled r is infinitely heavy, and cannot be moved.
- Then one ends up in a configuration which has 2 chips at the origin (one of which is r) and 1 chip each at positions

$$-\frac{n}{2}+1, \dots, -1, 1, \dots, \frac{n}{2}-1.$$



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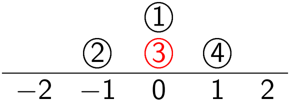

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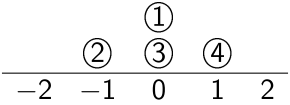

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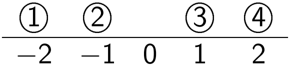

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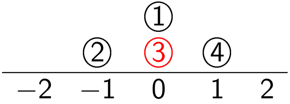
Motivation

- Consider the last stage where r is still infinitely heavy. E.g.



- That configuration can be considered as a permutation $\pi \in S_{n-1}$ plus an extra label, r.
- In the above example, $\pi = 213$, $r = 3$.
- According to HMP, all pairs (π, r) that arise this way end up sorted.
- It is natural to ask what are all the pairs which end up being sorted.

Notation

- Suppose $\pi = (\pi_1, \ldots, \pi_n) \in S_n$ and $r \in [n+1]$.
- Let $L_n = \{-\lfloor (n+1)/2 \rfloor, \ldots, -1, 0, 1, \ldots, \lfloor n/2 \rfloor + 1\}$.

1 Place the elements $\pi_1, \ldots, \pi_n$ in positions

$$-\left\lfloor \frac{n-1}{2} \right\rfloor, \ldots, -1, 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

2 Increase the labels in π greater than or equal to r by 1.

3 Add r to the origin.

- We will call this initial condition $\pi^{(r)}$.
- Eg with $r = 2$: $\rho = 3142 \in S_4$, $\sigma = 25134 \in S_5$.

$$\rho^{(2)} = \begin{array}{cccccc} & & 1 & & & \\ & 4 & {\color{red}2} & 5 & 3 & \\ \hline -2 & -1 & 0 & 1 & 2 & 3 \end{array}, \quad \sigma^{(2)} = \begin{array}{ccccccc} & & & 1 & & & \\ 3 & 6 & & {\color{red}2} & 4 & 5 & \\ \hline -3 & -2 & -1 & 0 & 1 & 2 & 3 \end{array}.$$

Definitions

- For $\pi \in S_n$ and $r \in [n+1]$, we consider the toppling dynamics.
- The toppling dynamical system on L_n can be considered as a map $T : S_n \times [n+1] \to S_{n+1}$.
- Let id be the identity (namely sorted) permutation.

Definition

We say that a permutation π is *r-toppleable* if $T(\pi, r) = \mathrm{id}$, and we say that π is *toppleable* if π is r-toppleable for all $r \in [n+1]$.

Basic properties

Proposition

Fix $\pi \in S_n$ and $r \in [n+1]$. The toppling dynamical system on L_n with initial condition $\pi^{(r)}$ satisfies the following properties.

1. The final configuration is deterministic.
2. At every time step, the configuration lives in L_n.
3. In the final configuration, there is precisely one chip at every position in L_n, except the origin (resp. position 1) when n is odd (resp. even).

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Main idea: No position contains more than 2 chips!

Number of toppleable permutations

- Let $t_r(n)$ be the number of r-toppleable permutations.
- Let $t(n)$ be the number of toppleable permutations in S_n.
- For $n = 3$, there are four 1-toppleable permutations, namely 123, 213, 132 and 231, . . .
- and four 4-toppleable permutations, namely 123, 213, 132 and 312.
- Therefore, $t_1(3) = t_4(3) = 4$.
- The common permutations among these turn out also to be 2- and 3-toppleable.
- Hence $t(3) = t_2(3) = t_3(3) = 3$.

Data

$n \setminus r$	1	2	3	4	5	6	7	8	9
3	4	3	3	4					
4	14	10	7	7	8				
5	46	38	31	31	38	46			
6	230	184	146	115	115	130	146		
7	1066	920	790	675	675	790	920	1066	
8	6902	5836	4916	4126	3451	3451	3842	4264	4718

The number of r-toppleable permutations, $t_r(n)$, for $3 \leq n \leq 8$.
The number of toppleable permutations are in red.
Note the symmetry for odd n.

Statement of Main Result 3

Statement of monotonicity theorem

Background for the results: excedance sets

- An excedance of a permutation π is any position i such that $\pi_i > i$.
- The positions at which there are excedances for π is called the excedance set of π.
- Ehrenborg and Steingrímsson (Adv. Appl. Math., 2000) initiated the study of permutations whose excedance set is $\{1, \ldots, k\}$ for $0 \leq k \leq n-1$.
- They gave a formula for the number $a_{n,k}$ of such permutations in S_n.
- One surprising result they found is that $a_{n,k} = a_{n,n-1-k}$.
- A related result of Clark and Ehrenborg (Europ. J of C, 2010) is

$$\sum_{r,s \geq 0} a_{r+s,s} \frac{x^r}{r!} \frac{y^s}{s!} = \frac{e^{-x-y}}{(e^{-x} + e^{-y} - 1)^2}.$$

Main result 1

Theorem (A., Hathcock and Tetali, 2020+)

For all n,

$$t(n) = t_{\lfloor n/2 \rfloor + 1}(n) = t_{\lfloor n/2 \rfloor + 2}(n).$$

Furthermore,

$$t(n) = a\left(n, \left\lfloor \frac{n-1}{2} \right\rfloor\right).$$

Using the exponential generating function, de Andrade, Lundberg and Nagle (Europ. J. of C, 2015) obtained the asymptotic formula,

$$t(n) = \frac{1}{2\log 2\sqrt{1-\log 2} + o(1)} \frac{n!}{(2\log 2)^n}.$$

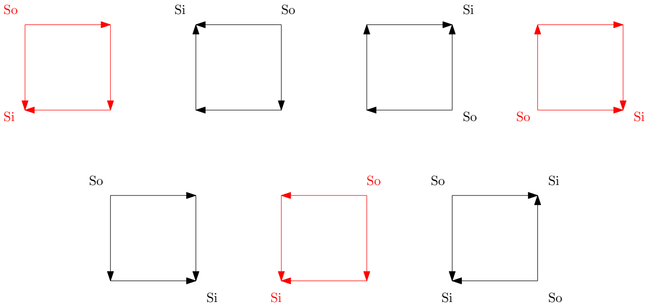
Acyclic orientations and chromatic polynomials

- Let G be a simple (no loops or multiple edges) undirected graph.
- An orientation of G is an assignment of arrows to the edges of G.
- An acyclic orientation (AO) is an orientation in which there is no directed cycle.
- A proper colouring of G is an assignment of colours to vertices such that no two adjacent vertices get the same colour.
- The chromatic polynomial of G, denoted $\chi_G(q)$, is the number of proper colourings of G with q colours.

Theorem (Stanley, *Disc. Math.*, 1973)

The number of acyclic orientations of G (up to sign) is $\chi_G(-1)$.

Example: C_4, the 4-cycle



There are 14 acyclic orientations for C_4. Seven are shown here. The other seven are obtained by reversing each of the arrows. The chromatic polynomial is $\chi_{C_4}(q) = q^4 - 4q^3 + 6q^2 - 3q$.

Acyclic orientations with unique sink

Definition

An acyclic orientation with a unique sink (AUSO) is an acyclic orientation with exactly one sink.

Theorem (Greene and Zaslavsky, *Trans. of the AMS*, 1983)

The number of AUSOs of G (up to sign) is independent of the sink and equal to (up to sign) the linear coefficient of $\chi_G(-1)$.

C_4 has 3 AUSOs, shown in red on the previous page.

Main result 2

Recall that $K_{m,n}$ is the complete bipartite graph with parts of size m and n.
For example, $C_4 \cong K_{2,2}$.

Theorem (A., Hathcock and Tetali, 2020+)

For all n, $t(n)$ is equal to the number of acyclic orientations with a fixed unique sink of $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor + 1}$.

This proof is bijective.

Poly-Bernoulli numbers

- The well-known polylogarithm function is given by

$$\mathrm{Li}_k(z) = \sum_{i=1}^{\infty} \frac{z^i}{i^k}.$$

- Recall that a position k is an ascent in a permutation if $\pi_k < \pi_{k+1}$.
- The Eulerian number $\left\langle {m \atop j} \right\rangle$ is the number of permutations in S_n with j ascents.
- For a non-negative integer m,

$$\mathrm{Li}_{-m}(z) = \frac{\sum_{j=0}^{m-1} \left\langle {m \atop j} \right\rangle z^{m-j}}{(1-z)^{m+1}}.$$

Poly-Bernoulli numbers

- Poly-Bernoulli numbers of type B were defined by Kaneko (1997) via the exponential generating function,

$$\sum_{n=0}^{\infty} B_{n,k} \frac{x^n}{n!} = \frac{\mathrm{Li}_{-k}(1-e^{-x})}{1-e^{-x}},$$

- A surprising result is that $B_{k,n} = B_{n,k}$.
- There are many combinatorial interpretations for $B_{n,k}$.
- A permutation $\pi \in S_{k+n}$ is said to be a (k,n)-Vesztergombi permutation if $-k \leq \pi_i - i \leq n$ for $1 \leq i \leq k+n$.
- The number of (k,n)-Vesztergombi permutations is $B_{n,k}$.

The first few poly-Bernoulli numbers of type B

$n \backslash k$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	4	8	16	32
2	1	4	14	46	146	454
3	1	8	46	230	1066	4718
4	1	16	146	1066	6906	41506
5	1	32	454	4718	41506	329462

Forward difference operators

- Let Δ be the discrete (forward) difference operator, i.e. for any function $f(n)$, $\Delta(f(n)) = f(n+1) - f(n)$.
- The higher difference operators are obtained by composition.
- For example, $\Delta^2(f(n)) = f(n+2) - 2f(n+1) + f(n)$.
- The sequence $\Delta^k(f)$ is also known as a binomial transform of the sequence f.
- Note that $\Delta^0(f(n)) = f(n)$.

Main result 3

Back to data

Theorem (A. and Bényi, 2021+)

The number of r-toppleable permutations in S_n is

$$t_r(n) = \Delta^{r-1}\left(B_{n-p+1-r,p}\right),$$

where $p = \lfloor (n+1)/2 \rfloor$ and Δ acts on the first index.

- We generalise this result to any position of adding the extra chip.
- We also characterise all possible final permutations and enumerate permutations toppling to these.

Focus on odd n

- For each statement, the results for odd and even n differ slightly.
- To make the presentation cleaner, we state the results only for odd n.
- This will avoid the presence of floors and ceilings all over the place.
- The corresponding results for even n are given in `arXiv:2010.11236`.

Monotonicity

Theorem (A., Hathcock and Tetali, 2020+)

Let $\pi \in S_{2m+1}$.

1. *Suppose $2 \leq r \leq m+1$. Then π is $(r-1)$-toppleable if π is r-toppleable.*
2. *Suppose $m+2 \leq r \leq 2m$. Then π is $(r+1)$-toppleable if π is r-toppleable.*
3. *π is $(m+1)$-toppleable if and only if π is $(m+2)$-toppleable.*

Back to data

The notion of a pass

- For $\pi \in S_{2m+1}$, let the number of chips at each site of L_n in $\pi^{(r)}$ be
$$p^{(r)} = (\_, 1, \dots, 1, \hat{2}, 1, \dots, 1, \_).$$
- Topple as follows:
$$\begin{aligned} p^{(r)} &\to (\_, 1, \dots, 1, 1, 2, \hat{\_}, 2, 1, 1, \dots, 1, \_) \\ &\to (\_, 1, \dots, 1, 2, \_, \hat{2}, \_, 2, 1, \dots, 1, \_) \\ &\to (\_, 1, \dots, 2, \_, 1, \hat{2}, 1, \_, 2, \dots, 1, \_). \end{aligned}$$
- At this point, we leave the origin unchanged and start to topple the vertices with 2 chips both on the left and right, until we reach the end.
- We then arrive at the configuration with chip counts given by
$$(1, \_, 1, \dots, 1, \hat{2}, 1, \dots, 1, \_, 1).$$

The notion of a pass

- Now, the extremal points cannot be modified by any further topplings and are fixed.
- We call this sequence of topplings the first pass.
- This consists of $2m + 1$ individual topplings.
- Similarly, the second pass will be initiated by toppling the origin in a similar way, and we will end up with

$$(1, 1, \_, 1, \ldots, 1, \hat{2}, 1, \ldots, 1, \_, 1, 1).$$

- Continue this way until the configuration stabilizes.
- If n is odd, then we see that after $(n + 1)/2$ passes, the configuration will freeze leaving the origin empty.

Observations about passes

- Every chip between vacancies topples at least once in every pass.
- If $\pi \in S_{2m+1}$ is toppleable, then for $1 \leq i \leq m+1$, i and $2m+2-i$ get fixed in their correct positions at the end of the i'th pass.
- For example:

$$\begin{array}{ccccccc} & & & 1 & & & \\ \_ & 3 & 6 & 2 & 4 & 5 & \_ \end{array} \xrightarrow[\text{pass}]{\text{first}} \begin{array}{ccccccc} & & & 2 & & & \\ 1 & \_ & 3 & 6 & 4 & \_ & 5 \end{array}$$

$$\xrightarrow[\text{pass}]{\text{second}} \begin{array}{ccccccc} & & & 3 & & & \\ 1 & 2 & \_ & 4 & \_ & 6 & 5 \end{array} \xrightarrow[\text{pass}]{\text{third}} \begin{array}{ccccccc} 1 & 2 & 3 & \_ & 4 & 6 & 5 \end{array}.$$

Structure theorem

Theorem (A., Hathcock and Tetali, 2020+)

A permutation $\pi \in S_{2m+1}$ *is* $(m+1)-$*toppleable if and only if*

$$\pi_i \leq m+i, \quad 1 \leq i \leq m,$$
$$\pi_i \geq i-m, \quad m+1 \leq i \leq 2m+1.$$

Equivalently,

$$\pi_i^{-1} \in \{1, \ldots, m+i\}, \quad 1 \leq i \leq m+1$$
$$\pi_i^{-1} \in \{i-m, \ldots, 2m+1\}, \quad m+2 \leq i \leq 2m+1.$$

Main idea: The notion of a pass and induction.

Bijection

Lemma

Permutations $\pi \in S_{2m+1}$ *such that* $\pi_i \leq m + i$ *for* $1 \leq i \leq m$ *and* $\pi_i \geq i - m$ *for* $m + 1 \leq i \leq 2m + 1$ *are in bijection with permutations in* S_{2m+1} *whose excedance set is* $\{1, \ldots, m\}$*.*

Proof idea.

$$(\pi_1, \ldots, \pi_m | \pi_{m+1}, \ldots, \pi_{2m+1}) \\ \to \sigma = 2m + 2 - (\pi_m, \ldots, \pi_1 | \pi_{2m+1}, \ldots, \pi_{m+1}).$$

□

Proof of main result 1

Proof.

- By the monotonicity result, we see that $\pi \in S_{2m+1}$ is toppleable if it is $(m+1)$-toppleable.
- According to the structure theorem, $\pi_i \leq m + i$ for $1 \leq i \leq m$ and $\pi_i \geq i - m$ for $m + 1 \leq i \leq 2m + 1$.
- Now, the previous lemma proves that the number of such permutations is $a_{2m+1,m}$ bijectively, completing the proof.

□

Back to HMP toppling

Theorem (Lemma 12, Hopkins, McConville and Propp)

Starting with n chips at the origin, the position of chip k lies between $-\lfloor (n+1-k)/2 \rfloor$ and $\lfloor k/2 \rfloor$ for $1 \leq k \leq n$ at all times.

- When n is odd, $n = 2m+1$, the final configuration will contain single chips in all positions $-m$ through m.
- We now apply this condition to count permutations arising from this condition switching positions from $[-m, m]$ to $[n]$.
- For n even, the only permutation that appears as a result of toppling is id.
- We also consider this case, although it is not directly relevant to the toppling problem.

Collapsed permutations

Definition

We say that a permutation $\pi \in S_n$ is collapsed if

$$\pi_k^{-1} \geq \begin{cases} \lceil k/2 \rceil & n \text{ odd}, \\ 1 + \lfloor k/2 \rfloor & n \text{ even} \end{cases} \quad \text{and} \quad \pi_k^{-1} \leq \lceil n/2 \rceil + \lfloor k/2 \rfloor.$$

Let G_n be the subset of collapsed permutations in S_n

- For $n = 2m + 1$,

i	1	2	3	...	$2m$	$2m+1$
Position of $i \geq$	1	1	2	...	m	$m+1$
Position of $i \leq$	$m+1$	$m+2$	$m+2$	...	$2m+1$	$2m+1$

- For example, $G_3 = \{123, 132, 213\}$ and $G_4 = \{1234, 1324\}$.

Seidel triangle for the Genocchi numbers

- To state our results, we recall a well-known combinatorial triangle.
- The Seidel triangle is the triangular sequence $S_{n,k}$ for $n \geq 1$ given by

$$\begin{aligned} S_{1,1} &= 1, \\ S_{n,k} &= 0, \quad k < 2 \text{ or } (n+3)/2 < k, \\ S_{2n,k} &= \sum_{i \geq k} S_{2n-1,i}, \\ S_{2n+1,k} &= \sum_{i \leq k} S_{2n,i}. \end{aligned}$$

First few rows

$n \backslash k$	2	3	4	5	6
1	1				
2	1				
3	1	1			
4	2	1			
5	2	3	3		
6	8	6	3		
7	8	14	17	17	
8	56	48	34	17	
9	56	104	138	155	155
10	608	552	448	310	155

Genocchi numbers of the first kind

- The numbers on the rightmost diagonal are the Genocchi numbers of the first kind, g_{2n}.
- They counts permutations in S_{2n-3} whose excedence set is $\{1, 3, \ldots, 2n-5\}$.
- For example, $g_8 = 17$:

$$21435, 21534, 21543, 31425, 315, 24, 31542, 32415, 32514,$$
$$32541, 41523, 41532, 42513, 42531, 51423, 51432, 52413, 52431.$$

- The exponential generating function of g_{2n} is given by

$$\sum_{n\geq 0} g_{2n} \frac{x^{2n}}{(2n)!} = x \tan\left(\frac{x}{2}\right).$$

Odd collapsed permutations

Theorem

The number of collapsed permutations in S_{2n+1} is g_{2n+4}.

- Define a bijection $f : G_{2n+1} \to S_{2n+1}$ which send

$$\pi \mapsto \sigma = (\sigma_1, \ldots, \sigma_{2n+1})$$

such that

1. $\sigma_{2i} = \pi_i, \sigma_{2i-1} = \pi_{n+1+i}$ for $1 \le i \le n$, and
2. $\sigma_{2n+1} = \pi_{n+1}$.

- The bijection for $n = 1$ is illustrated below:

G_3	S_3 with excedence set $\{1\}$
132	213
123	312
213	321

Genocchi numbers of the second kind

- The numbers on the leftmost diagonal are the median Genocchi numbers or Genocchi numbers of the second kind, H_{2n+1}.
- They count among other things, ordered pairs $((a_1, \ldots, a_{n-1}), (b_1, \ldots, b_{n-1})) \in \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$ such that $0 \le a_k \le k$ and $1 \le b_k \le k$ for all k and $\{a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}\} = [n-1]$.
- For example, $H_7 = 8$:

$$((0,0),(1,2)),\ ((0,1),(1,2)),\ ((0,2),(1,1)),\ ((0,2),(1,2)),$$
$$((1,0),(1,2)),\ ((1,1),(1,2)),\ ((1,2),(1,1)),\ ((1,2),(1,2)).$$

- In terms of the Genocchi numbers of the first kind, we have

$$H_{2n+1} = \sum_{i=0}^{n} g_{2n-2i} \binom{n}{2i+1}.$$

Normalized median Genocchi numbers

- Although it is not clear either from the above definition or the formula, H_{2n+1} is always divisible by 2^n.
- The numbers $h_n = H_{2n+1}/2^n$ are called the normalized median Genocchi numbers.

  The first few numbers of this sequence are

$$\{h_n\}_{n=0}^{7} = \{1, 1, 2, 7, 38, 295, 3098, 42271\}.$$

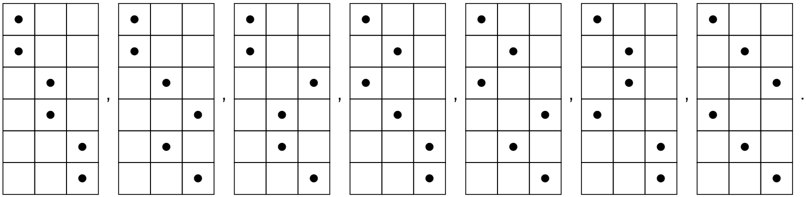
- A classical combinatorial interpretation for these are certain configurations first defined by Hippolyte Dellac in 1900.

Dellac configuration

Definition

A Dellac configuration of order n is a $2n \times n$ array containing $2n$ points, such that every row has a point, every column has two points, and the points in column j lie between rows j and $n + j$, both inclusive, $1 \leq j \leq n$.

For example, when $n = 3$, the 7 Dellac configurations are



Even collapsed permutations

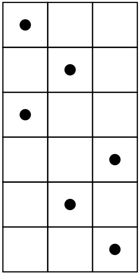
Theorem

The number of collapsed permutations in S_{2n} is given by H_{2n-1}.

- Both $2i$ and $2i+1$ have to lie in positions between $i+1$ and $i+n$, both inclusive, for $1 \leq i \leq n-1$.
- Thus, $\#G_{2n}$ is divisible by 2^{n-1}.
- Focus on $\pi \in G_{2n}$ such that $2i$ precedes $2i+1$ in one-line notation for all i.
- Since $\pi_1 = 1$ and $\pi_{2n} = 2n$ are forced, we focus on $(\pi_2, \ldots, \pi_{2n-1})$.

Bijection

- Construct a configuration C of points on an $(2n-2)\times(n-1)$ array as follows:
- For $2 \leq i \leq 2n-1$, place a point in position $(i-1, \lfloor \pi_i/2 \rfloor)$.
- C is a Dellac configuration and this can be inverted.
- For example, the permutation $1\underbrace{243657}8 \in G_8$ is in bijection with



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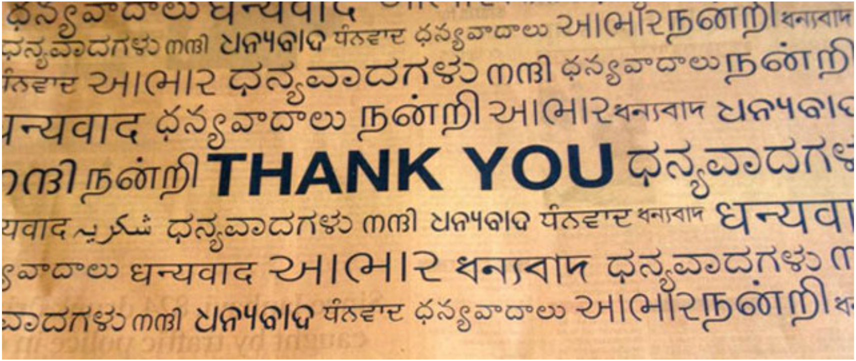
THANK YOU