Rigid Motion Invariants of Curves through Iterated-Integrals

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with

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Overview

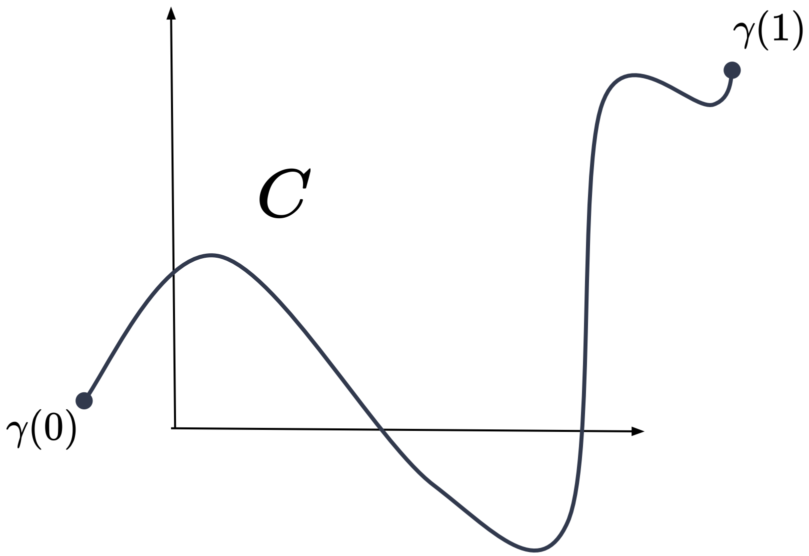
- What/Why iterated-integrals of curves?
- Invariantization via cross-sections
- Orthogonal action on iterated-integrals
- Some examples

Iterated-integrals of curves

- Consider a parameterized *path*

$$C : [0,1] \to \mathbb{R}^2$$

$$\gamma(t) = (x(t), y(t))$$



Iterated-integrals of curves

- Consider a parameterized *path*
- Want
  - Geometrically relevant features of C
- Why?
  - C represents some continuous sequential data
  - Finite-dim useful for machine learning
  - Shape Analysis, Human Activity Recognition

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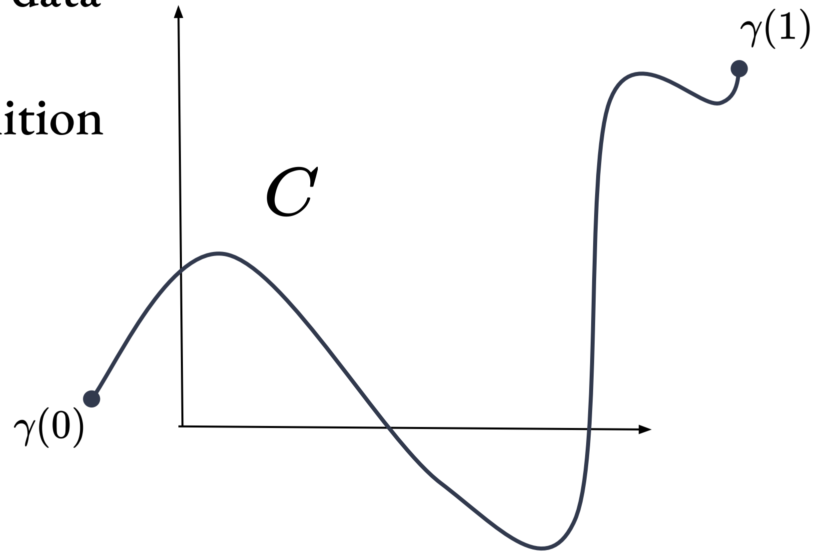

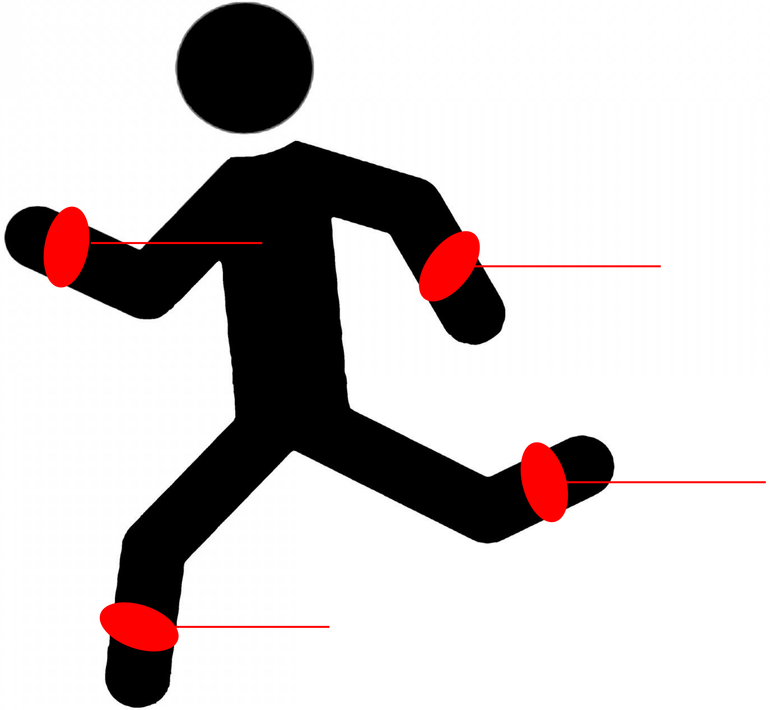

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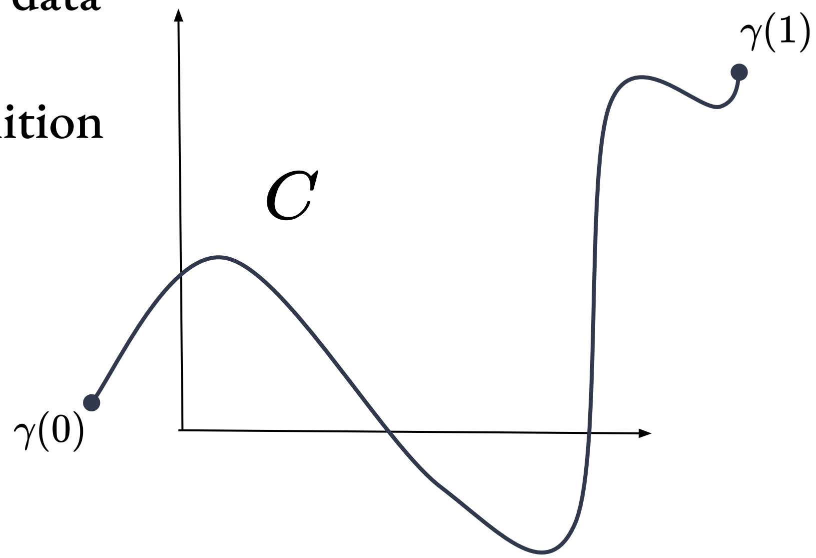

A primer on the signature method in machine learning
Ilya Chevyrev, Andrey Kormilitzin (2016)

Iterated-integrals of curves

- Iterated-integrals of the path

$$C : [0, 1] \to \mathbb{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

$$\int_0^1 dx(t)$$

$$\int_0^1 dy(t)$$

$$\int_0^1 \int_0^r dx(t)dy(r)$$

$$\int_0^1 \int_0^r dy(t)dx(r)$$

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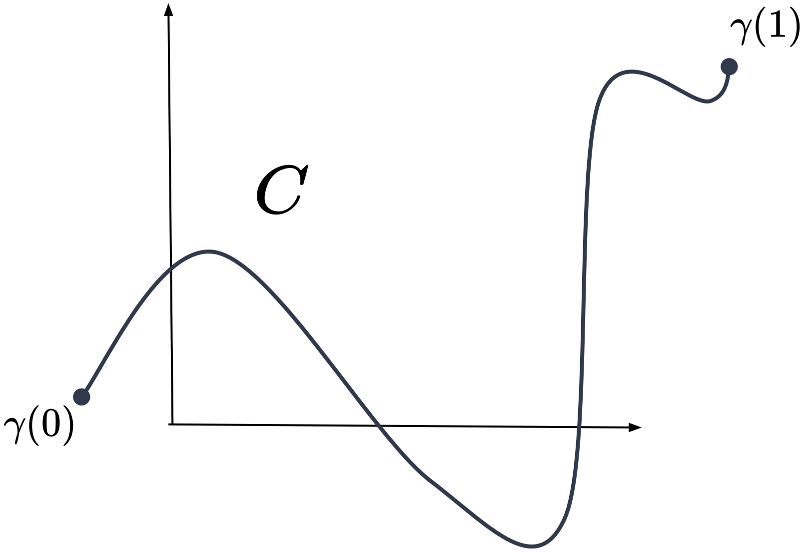
$$\vdots$$



Iterated-integrals of curves

- Iterated-integrals of the path
- Iterated-integral signature

$$IIS(C) = (1, 2, 12, 21, 11, 22, 111, \ldots)$$

$$C : [0,1] \to \mathbb{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

$$\int_0^1 dx(t) \longleftarrow 1$$

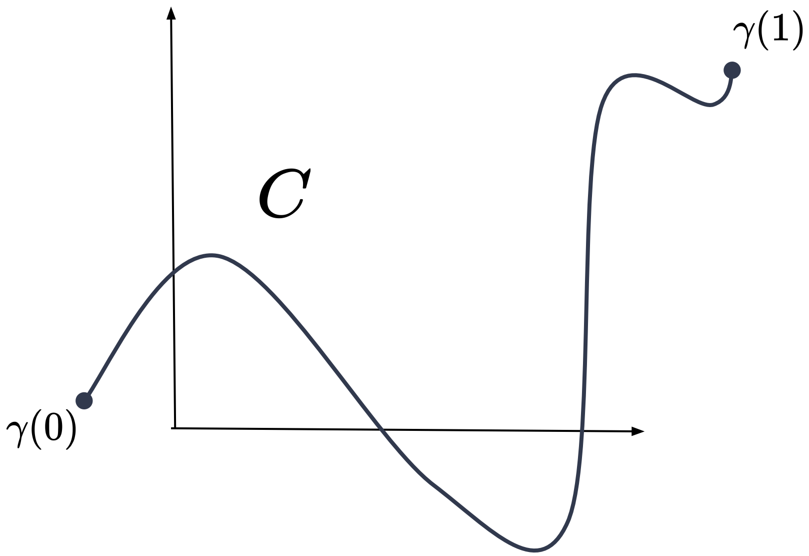
$$\int_0^1 dy(t) \longleftarrow 2$$

$$\int_0^1 \int_0^r dx(t)dy(r) \longleftarrow 12$$

$$\int_0^1 \int_0^r dy(t)dx(r) \longleftarrow 21$$

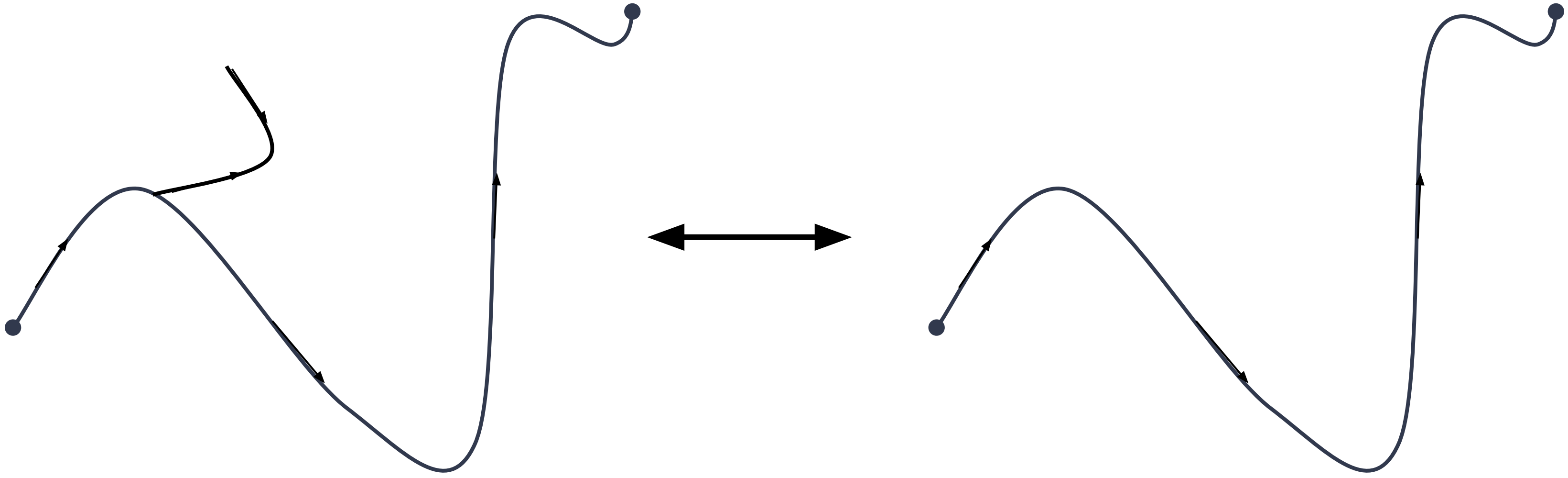
$$\int_0^1 \int_0^r dx(t)dx(r) \longleftarrow 11$$

$$\vdots$$



Why?

Theorem (Chen 54)
Two smooth paths have the same iterated-integral signature if and only if they are equal (up to tree-like extensions and translations).



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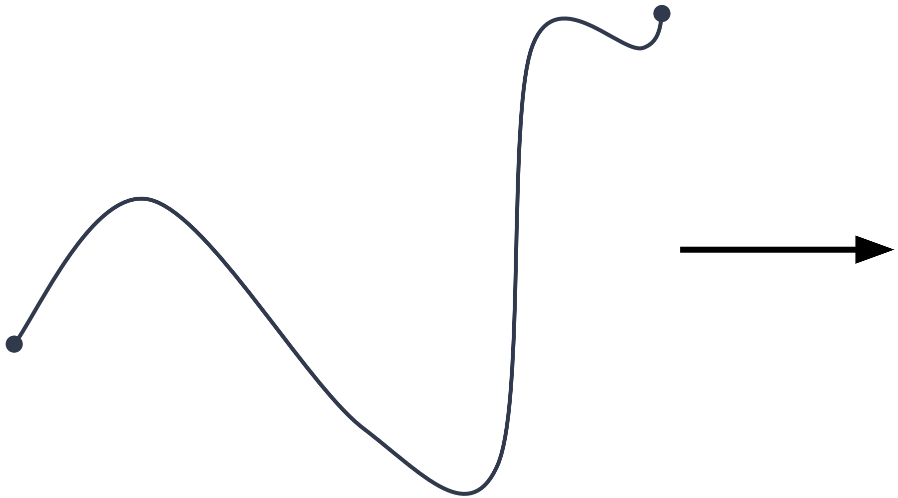

$$IIS(C)^{(2)} = (1, 2, 12, 21, 11, 22)$$

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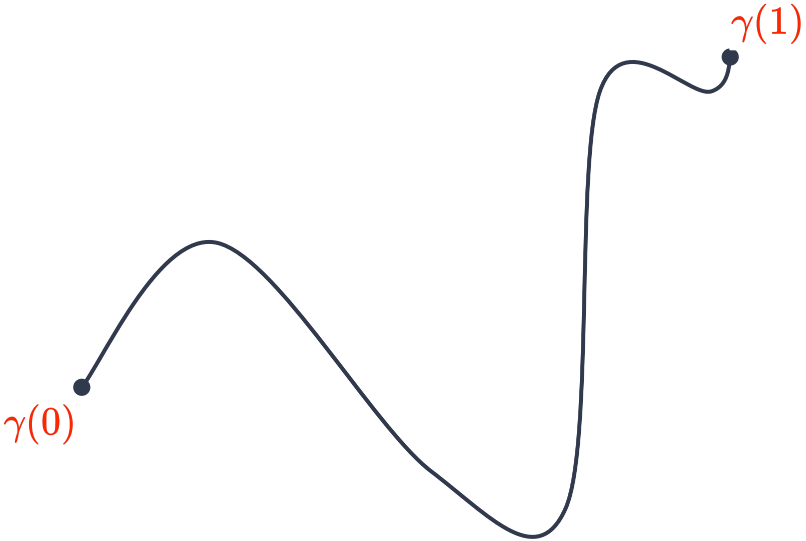

$$x(1) - x(0) \qquad\qquad y(1) - y(0)$$

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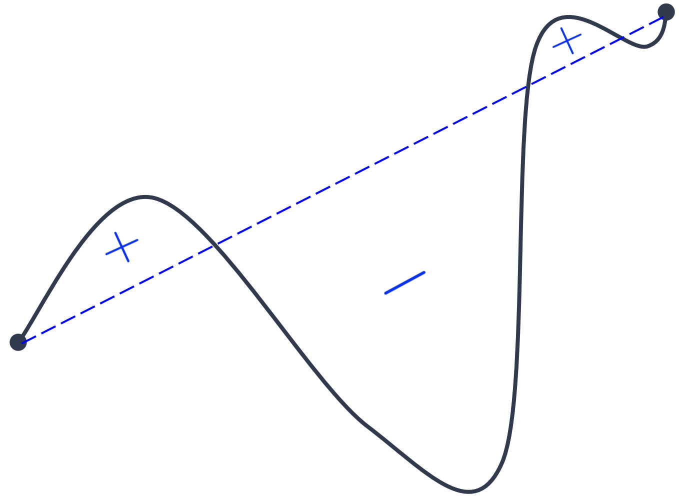

$$(1/2)(12 - 21)$$

$$IIS(C)^{(2)} = (1, 2, 12, 21, 11, 22)$$

Invariants

- Note that the previous functions were *Euclidean invariants*.
- Invariants are nice for shape analysis, human activity recognition, etc.
- What does the space of iterated-integral invariants look like?

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Order 2

$$\begin{aligned} &\mathbf{11} + \mathbf{22} \\ -&\mathbf{12} + \mathbf{21} \end{aligned}$$

Order 4

$$\begin{aligned}
&\mathbf{1111} - \mathbf{1122} + \mathbf{1212} + \mathbf{1221} + \mathbf{2112} + \mathbf{2121} - \mathbf{2211} + \mathbf{2222} \\
-&\mathbf{1112} - \mathbf{1121} + \mathbf{1211} - \mathbf{1222} + \mathbf{2111} - \mathbf{2122} + \mathbf{2212} + \mathbf{2221} \\
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Shuffle relation

$$1 \times 12 = 112 + 112 + 121$$

Invariants

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- Goals
    - Describe a minimal, functionally-independent set of invariants for each truncation level of the IIS.
    - Characterize the equivalence class of a curve's IIS

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- Goals (Orthogonal action: Rotations + Reflections)
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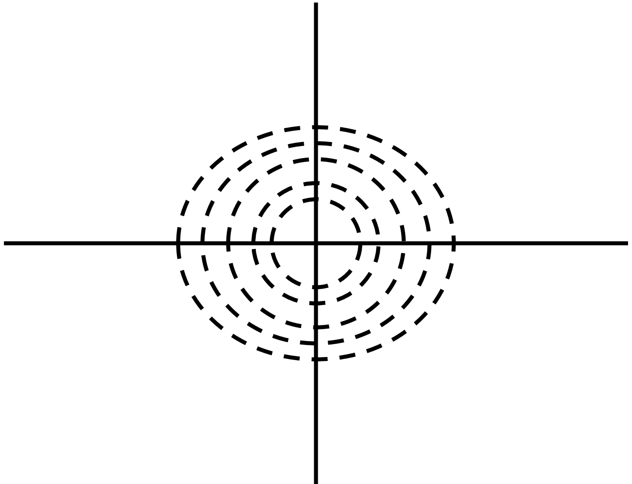
Cross-sections and Moving Frame

- We can accomplish this goal using the Moving Frame Method (Fels, Olver 99)



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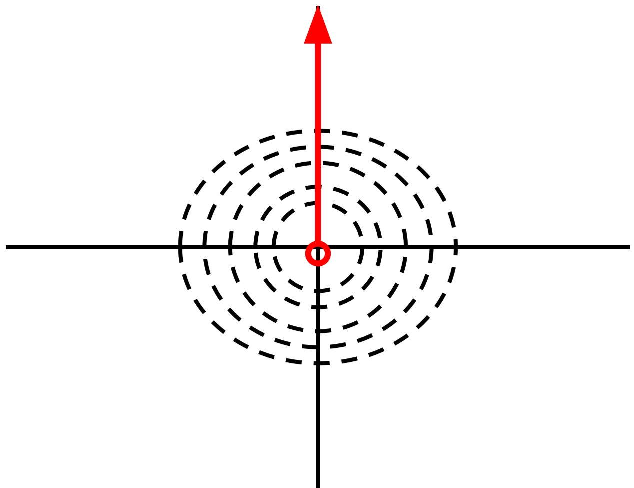

Cross-sections and Moving Frame

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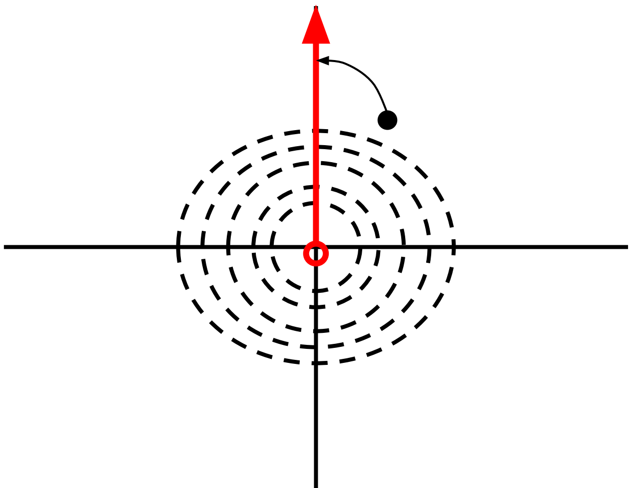
- Moving Frame $\rho : \mathbb{R}^2 \to \mathcal{O}_2$
  - Group element taking a point to the cross section

$$x \to \left(\frac{y}{\sqrt{x^2+y^2}}\right) x + \left(\frac{-x}{\sqrt{x^2+y^2}}\right) y$$

$$y \to \left(\frac{x}{\sqrt{x^2+y^2}}\right) x + \left(\frac{y}{\sqrt{x^2+y^2}}\right) y$$



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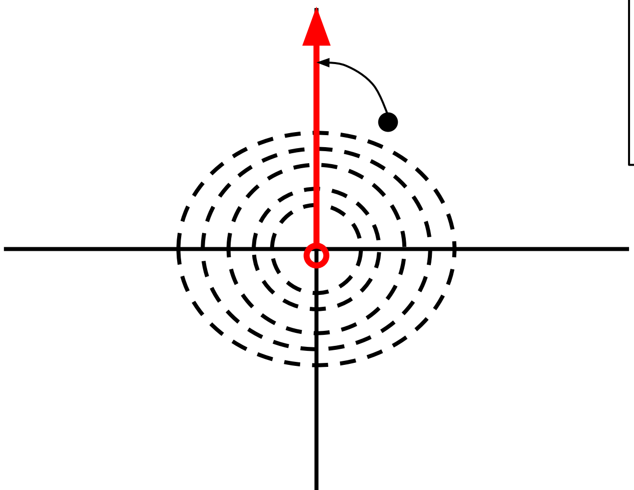
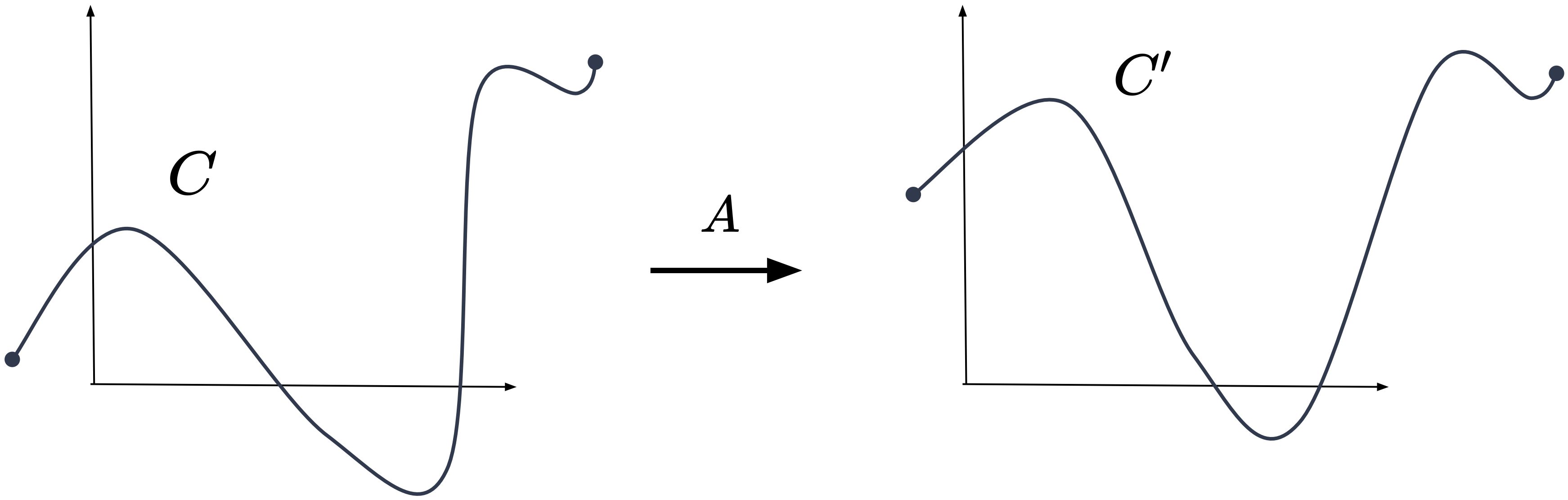


Two points are equivalent if and only if they have the same cross-section representative.

Action on the IIS

- Consider the action of $A \in \mathcal{O}_d$ on $\mathbb{R}^d$



Action on the IIS

- Consider the action of $A \in \mathcal{O}_d$ on $\mathbb{R}^d$
- Induces a *joint* action on *IIS(C)*

$$A \cdot IIS(C) := IIS(A \cdot C)$$

$$\begin{aligned}
A \cdot 1 &= A(1, 2, \ldots, d) \\
A \cdot 2 &= A(1, 2, \ldots, d) \\
&\vdots \\
A \cdot d &= A(1, 2, \ldots, d) \\
A \cdot 11 &= A(11, 12, 13, \ldots, 1d, \ldots, d1) \\
&\vdots
\end{aligned}$$

Action on the IIS

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Relationships between entries (shuffle relations)!

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Log-signature Transform

- Log-signature map: bijection from space of iterated-integral signatures

$$IIS(C) = (1, 2, 11, 12, 21, 22, \ldots)$$

$$\downarrow$$

$$\log IIS(C) = (c_1, c_2, c_{12}, \ldots)$$

Log-signature Transform

$$c_{12} = [1,2] = 12 - 21$$

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$$A \cdot C = \tilde{C}$$

$$A \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{bmatrix} = \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_d \end{bmatrix}$$

$$A \begin{bmatrix} 0 & c_{12} & c_{13} & \ldots & c_{1d} \\ -c_{12} & 0 & c_{23} & \ldots & c_{2d} \\ -c_{13} & -c_{23} & 0 & \ldots & c_{3d} \\ \vdots & & & & \vdots \\ -c_{1d} & -c_{2d} & -c_{3d} & \ldots & 0 \end{bmatrix} A^T = \begin{bmatrix} 0 & \tilde{c}_{12} & \tilde{c}_{13} & \ldots & \tilde{c}_{1d} \\ -\tilde{c}_{12} & 0 & \tilde{c}_{23} & \ldots & \tilde{c}_{2d} \\ -\tilde{c}_{13} & -\tilde{c}_{23} & 0 & \ldots & \tilde{c}_{3d} \\ \vdots & & & & \vdots \\ -\tilde{c}_{1d} & -\tilde{c}_{2d} & -\tilde{c}_{3d} & \ldots & 0 \end{bmatrix}$$

Log-signature Transform

- Cross section on log *IIS(C)* equivalent to cross-section on $\mathbb{R}^d \bigoplus \mathfrak{so}_d(\mathbb{R})$

$$A \cdot (v, M) \to (Av, AMA^T)$$

Log-signature Transform

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1. Iteratively construct a *relative section* over $\mathbb{C}^d \bigoplus \mathfrak{so}_d(\mathbb{C})$
2. Show this induces a cross-section over $\mathbb{R}^d \bigoplus \mathfrak{so}_d(\mathbb{R})$(for most curves)

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$$\mathcal{K} = \{c_i = 0, c_{j(i+1)} = 0, c_d > 0, c_{i(i+1)} > 0 \,|\, 1 \leq i \leq d-1, 1 \leq j < i\}$$

Log-signature Transform

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$$\mathcal{K}_1 = \{c_i = 0, c_d > 0 \,|\, 1 \leq i \leq d-1\}$$

$$\mathcal{K}_2 = \{c_i = 0, c_d > 0, c_{1d} = c_{2d} = \cdots = c_{(d-2)d} = 0, c_{(d-1)d} > 0 \,|\, 1 \leq i \leq d-1\}$$

$$\vdots$$

$$(v, M) = \left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \tilde{c}_d \end{bmatrix}, \begin{bmatrix} 0 & \tilde{c}_{12} & 0 & \dots & 0 \\ -\tilde{c}_{12} & 0 & \tilde{c}_{23} & \dots & 0 \\ 0 & -\tilde{c}_{23} & 0 & \ddots & 0 \\ \vdots & & & & \tilde{c}_{(d-1)d} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \right),$$

Why?

Theorem (Diehl, Preiß, R., Tapia 20)
Two smooth paths are equivalent up to translations, rotations, and reflections (and tree-like extensions) if and only if their log-signatures have the same value on the cross-section $\mathcal{K}$

Why?

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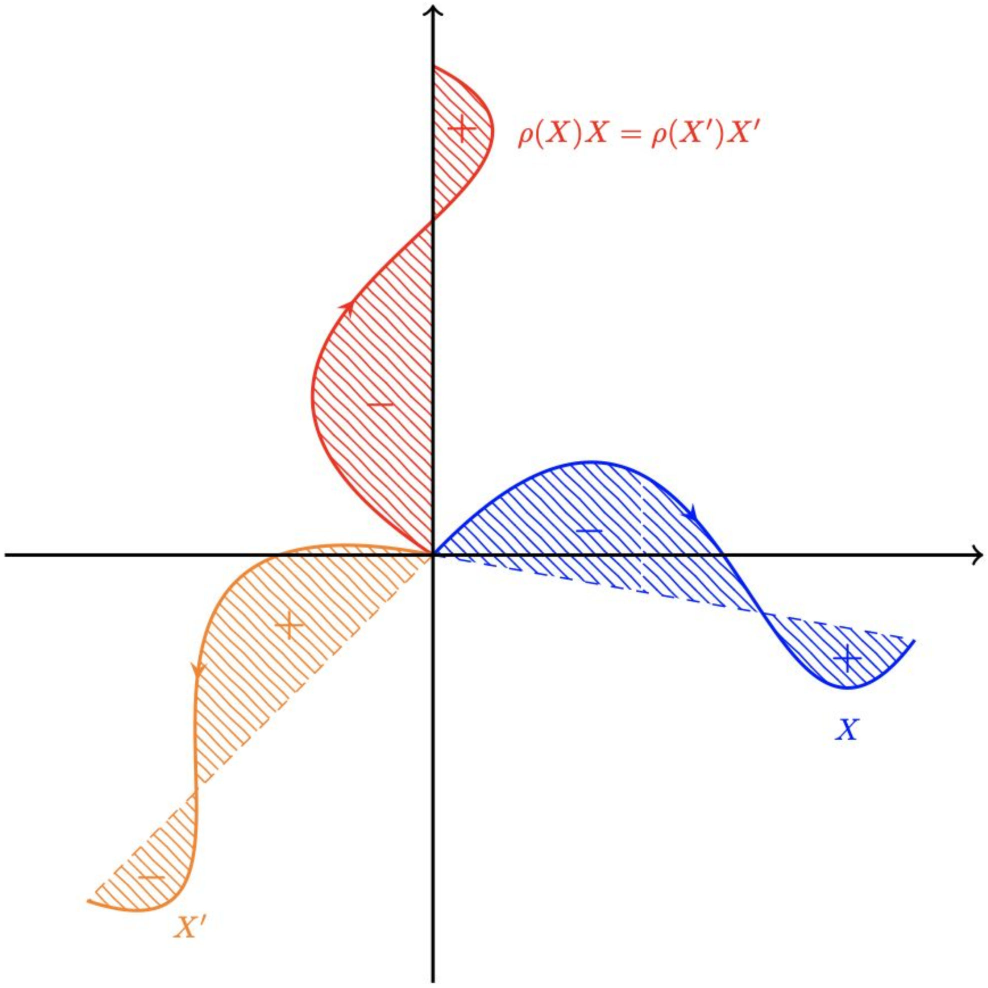
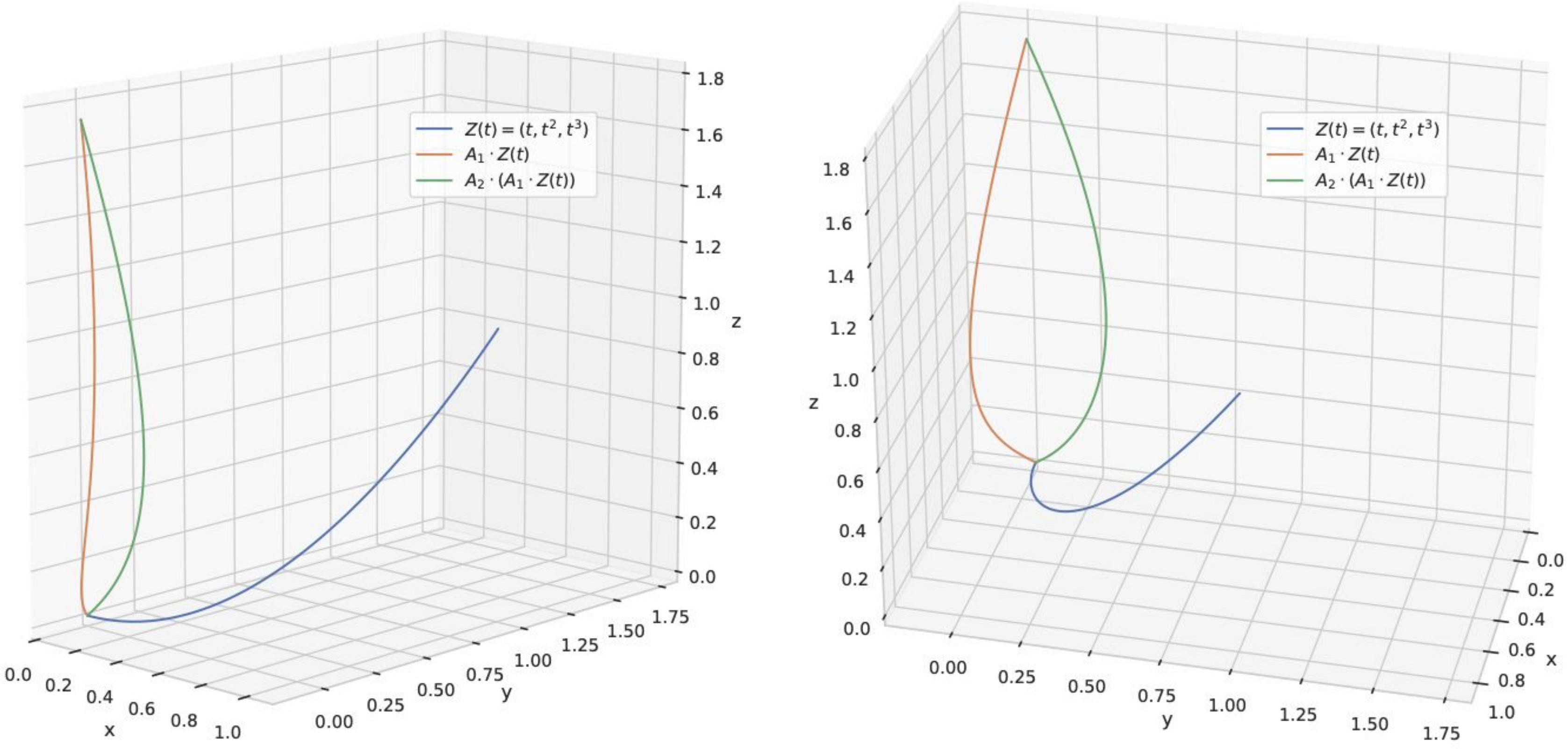
Two smooth paths are equivalent up to translations, rotations, and reflections (and tree-like extensions) if and only if their log-signatures have the same value on the cross-section $\mathcal{K}$

Theorem (Diehl, Preiß, R., Tapia 20)
Two smooth paths have equivalent truncated (of order k) iterated-integral signatures under translations, rotations, and reflections (and tree-like extensions) if and only if their log-signatures up to order k have the same value on the cross-section $\mathcal{K}$

- Cross-section characterizes equivalence classes of truncated IIS
- Gives an explicit method for vectorizing then invariantizing a curve.
- Don't need to compute complicated invariants for high orders.



$\rho(X)X = \rho(X')X'$
$+$
$-$
$-$
$+$
X
$+$
$-$
X'


$Z(t) = (t, t^2, t^3)$
$A_1 \cdot Z(t)$
$A_2 \cdot (A_1 \cdot Z(t))$
x
y
z
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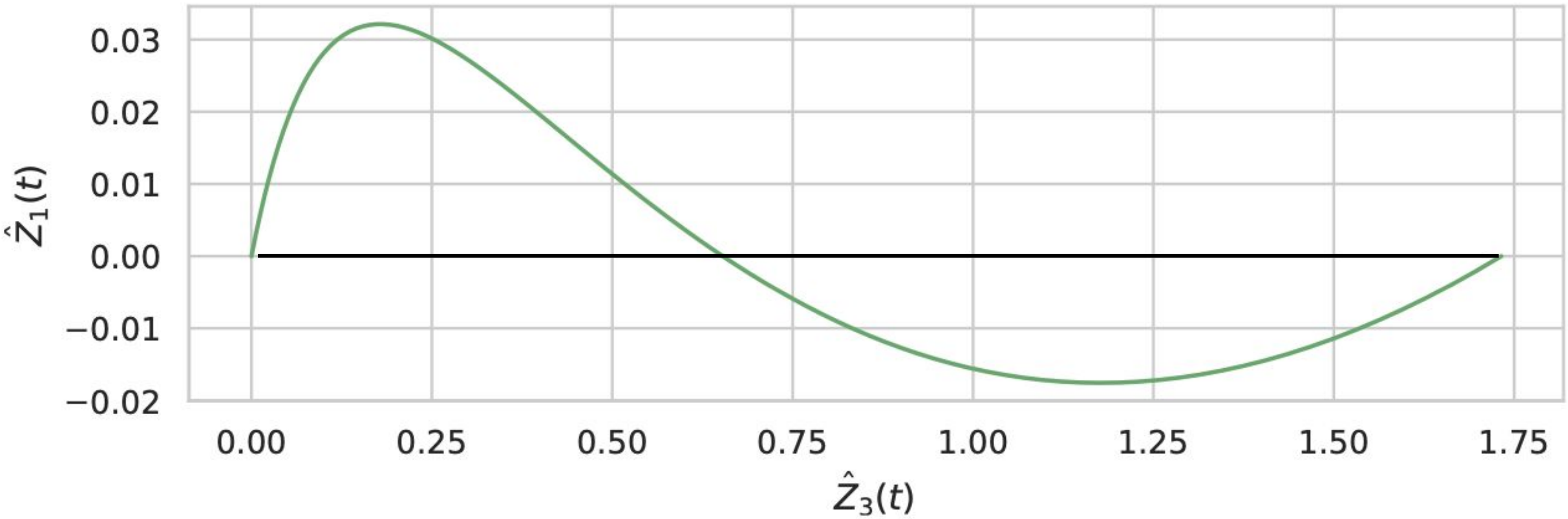
$c_{13} = 0$



0.03
0.02
0.01
0.00
−0.01
−0.02
$\hat{Z}_1(t)$
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
$\hat{Z}_3(t)$

What Next?

- How well do these invariantized features perform in practice?
- Other Group Actions

Thank you!