Existence of Mean Curvature Flow Singularities with Bounded Mean Curvature

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Introduction & Motivation

Definition
A family of embeddings $\mathbf{F}(t) : \Sigma^{N-1} \to \Sigma^{N-1}(t) \subset \mathbb{R}^N$ moves by **mean curvature flow** if

$$\partial_t \mathbf{F} = H\nu \tag{MCF}$$

where

$$H = \text{the mean curvature of } \Sigma(t) \text{ at } \mathbf{F}(\cdot, t)$$
$$\nu = \text{the inner unit normal vector to } \Sigma(t) \text{ at } \mathbf{F}(\cdot, t)$$

Question
When exactly do singularities occur?

Blow Up at Finite Time Singularities

Theorem (Huisken '84)
If the mean curvature flow of closed hypersurfaces $\Sigma(t)$ becomes singular at time $T < \infty$, then the second fundamental form $A_{\Sigma(t)}$ blows up at time T,

$$\textit{i.e.} \qquad \limsup_{t \nearrow T} \sup_{\Sigma(t)} |A| = \infty$$

Question
Does the mean curvature $H = \operatorname{tr} A$ always blow up at a finite time singularity?
Equivalently, if $\sup_{[0,T)} \sup_{\Sigma(t)} |H| < \infty$, can we smoothly extend the flow to $[0, T + \epsilon)$?

Earlier Results

H blows up at the first singular time $T < \infty$ if ...

- (Huisken–Sinestrari '99) $\Sigma(0)$ is mean convex $H_{\Sigma(0)} \geq 0$,
- (Le–Šešum '10 ($\epsilon = 1/2$), Cooper '11)

$$\sup_{\Sigma(t)} |A| \leq \frac{C}{(T-t)^{1-\epsilon}}$$

- (Lin–Šešum '16)

$$\int_{\Sigma(0)} \left| A - \frac{1}{N-1} Hg \right|^2 < \epsilon, \qquad \text{or}$$

- (H. Li–B. Wang '19) $N = 3$.

(Le–Šešum, Xu–Ye–Zhao, $\ldots$) Progress on extension problem.

Main Theorem

Theorem (S– '20)
For any $N \geq 8$, there exists a smooth, properly embedded mean curvature flow solution $\left\{\Sigma^{N-1}(t) \subset \mathbb{R}^N\right\}_{t\in[0,T)}$ that becomes singular at $T < \infty$ with

$$\limsup_{t\nearrow T} \sup_{\Sigma(t)} |A| = \infty \qquad \textit{and} \qquad \sup_{t\in[0,T)} \sup_{\Sigma(t)} |H| < \infty.$$

Velázquez's Mean Curvature Flow Solution

Theorem (Velázquez '94)
Let $n \geq 4$ and $k \geq 2$. There exists a smooth, properly embedded mean curvature flow solution $\Sigma_k^{2n-1}(t) \subset \mathbb{R}^n \times \mathbb{R}^n$ which:

- *is $O(n) \times O(n)$-invariant,*
- *becomes singular at $\mathbf{0} \in \mathbb{R}^{2n}$ and $T < \infty$,*

$$\frac{1}{\sqrt{T-t}}\Sigma_k(t) \xrightarrow[t \nearrow T]{\text{good estimates}} \mathcal{C}, \quad \text{and}$$

$$\frac{1}{(T-t)^{\sigma+\frac{1}{2}}}\Sigma_k(t) \xrightarrow[t \nearrow T]{\text{coarser estimates}} \overline{\Sigma},$$

where

- $\mathcal{C}$ = minimal Simons cone,
- $\sigma = \sigma_{n,k} > 0$, and
- $\overline{\Sigma}$ = smooth minimal surface desingularizing $\mathcal{C}$ (Alencar '93).

Strategy for Proof of Main Theorem

Vanishing theorems
& Rescaling Argument
$\implies$ H bounded at small scales near the singularity
(cf. Brendle-Kapouleas '16, Bamler-Kleiner '17)

Interior Estimates (Ecker-Huisken '91)
& Sub-/Super-solutions
$\implies$ H bounded elsewhere

Vanishing Theorems

Vanishing Theorem for $\overline{\Sigma}$

The linearization of H_Σ at a minimal surface Σ is the *Jacobi operator* $\Delta_\Sigma + |A_\Sigma|^2$.

Theorem (Vanishing Theorem for $\overline{\Sigma}$)
Let $n \geq 4$ and $\overline{\Sigma}^{2n-1}$ be the smooth minimal surface desingularizing the Simons cone.
If $u(|x|, t)$ is a smooth solution of

$$\partial_t u = \Delta_{\overline{\Sigma}} u + |A_{\overline{\Sigma}}|^2 u \qquad \text{on } \overline{\Sigma} \times (-\infty, T]$$

and, for some $C_0 > 0$ and $a > |\alpha(n)|$,

$$|u(|x|, t)| \leq \frac{C_0}{(1 + |x|)^a}$$

then $u \equiv 0$.

Vanishing Theorem for $\mathcal{C}$

For $\Sigma = \mathcal{C}$ the minimal Simons' cone, the Jacobi operator

$$\partial_t u = \Delta_{\mathcal{C}} u + |A_{\mathcal{C}}|^2 u$$

applied to $u = u(|x|, t)$ becomes the parabolic Euler equation

$$\partial_t u = \frac{1}{2}\partial_{rr} u + \frac{n-1}{r}\partial_r u + \frac{n-1}{r^2} u \qquad (r = |x| \in (0, \infty))$$

Theorem (Vanishing Theorem for $\mathcal{C}$)
Let $n \geq 4$. If $u(|x|, t)$ solves

$$\partial_t u = \Delta_{\mathcal{C}} u + |A_{\mathcal{C}}|^2 u \qquad \text{on } \mathcal{C}^{2n-1} \times (-\infty, T]$$

and, for some $C_0 > 0$ and $|\alpha(n)| < a < |\alpha(n)| + 1$,

$$|u(|x|, t)| \leq \frac{C_0}{|x|^a}$$

then $u \equiv 0$.

Rescaling Argument

Let $\Lambda(t) = \Lambda_{n,k}(t)$ denote the blow-up rate of the Velázquez mean curvature flow solution $\Sigma_k^{2n-1}(t)$.

$$\sup_{\Sigma(t)} |A| \sim \Lambda(t) = (T-t)^{-\sigma_{n,k}-\frac{1}{2}} \gg \frac{1}{\sqrt{T-t}} \gg 1$$

Theorem

If $|\alpha(n)| < a < |\alpha(n)| + 1$ and k is large enough so that $\lambda_k \left(1 - \frac{a}{|\alpha|+1}\right) - \frac{1}{2} \geq 0$, then the Velázquez mean curvature flow solution $\Sigma_k^{2n-1}(t)$ satisfies

$$\begin{aligned} &\sup_{t\in[0,T)} \sup_{|x|\leq\sqrt{T-t}} |H_{\Sigma(t)}(x)| \\ \leq &\sup_{t\in[0,T)} \sup_{|x|\leq\sqrt{T-t}} (1+\Lambda(t)|x|)^a \, |H_{\Sigma(t)}(x)| \\ < &\infty. \end{aligned}$$

Proof.
Suppose not. Then there exists

$$t_i \nearrow T \qquad x_i \in \overline{B\left(\sqrt{T - t_i}\right)}$$

$$(1 + \Lambda_i |x_i|)^a |H(x_i)| = \sup_{t \le t_i} \sup_{|x| \le \sqrt{T-t}} (1 + \Lambda(t)|x|)^a |H(x)| \doteqdot M_i \nearrow +\infty$$

where $\Lambda_i \doteqdot \Lambda(t_i)$.

Case 1: $|x_i| \lesssim \Lambda_i^{-1}$

Define rescaled mean curvature flows

$$\tilde{\Sigma}_i(\tau) \doteqdot \Lambda_i \Sigma\left(t_i + \frac{\tau}{\Lambda_i^2}\right)$$

and let

$$\tilde{u}_i : \tilde{\Sigma}_i(\tau) \to \mathbb{R} \qquad \tilde{u}_i(\xi, \tau) \doteqdot \frac{\Lambda_i}{M_i} H_{\tilde{\Sigma}_i(\tau)}(\xi)$$

Then

$$\partial_\tau \tilde{u}_i = \Delta_{\tilde{\Sigma}_i(\tau)} \tilde{u}_i + |A_{\tilde{\Sigma}_i(\tau)}|^2 \tilde{u}_i \qquad (\tau \in [-t_i \Lambda_i^2, 0])$$

and

$$\left(1 + \frac{\Lambda(t_i + \tau/\Lambda_i^2)}{\Lambda_i} |\xi|\right)^a |\tilde{u}_i(\xi, \tau)| \leq 1$$

with equality at $\tau = 0$ and $\xi_i = x_i \Lambda_i$.

Passing to a subsequential limit $i \to \infty$

$$\tilde{\Sigma}_i \to \overline{\Sigma} \qquad \xi_i \to \xi_\infty \qquad \tilde{u}_i \to \tilde{u}_\infty$$

$$\partial_\tau \tilde{u}_\infty = \Delta_{\overline{\Sigma}} \tilde{u}_\infty + |A_{\overline{\Sigma}}|^2 \tilde{u}_\infty \qquad \text{on } \overline{\Sigma} \times (-\infty, 0]$$

and $|\tilde{u}_\infty(\xi, \tau)| \leq \dfrac{1}{(1 + |\xi|)^a}$ with equality at $(\xi_\infty, 0)$.

This contradicts the vanishing theorem for $\overline{\Sigma}$.

Case 2: $\Lambda_i^{-1} \ll |x_i| \ll \sqrt{T - t_i}$

Define rescaled mean curvature flows

$$\tilde{\Sigma}_i \doteqdot |x_i|^{-1} \Sigma(t_i + \tau |x_i|^2)$$

and let

$$\tilde{u}_i : \tilde{\Sigma}_i(\tau) \to \mathbb{R} \qquad \tilde{u}_i(\xi, \tau) \doteqdot \frac{\Lambda_i^a |x_i|^{a-1}}{M_i} H_{\tilde{\Sigma}_i(\tau)}(\xi)$$

Then

$$\partial_\tau \tilde{u}_i = \Delta_{\tilde{\Sigma}_i(\tau)} \tilde{u}_i + |A_{\tilde{\Sigma}_i(\tau)}|^2 \tilde{u}_i \qquad (\tau \in [-t_i |x_i|^{-2}, 0])$$

and

$$\left(\frac{\Lambda(t_i + \tau |x_i|^2)}{\Lambda_i} \right)^a |\xi|^a |\tilde{u}_i(\xi, \tau)| \leq 1$$

with equality at $\tau = 0$ and $\xi_i = \frac{x_i}{|x_i|}$.

Passing to a subsequential limit as $i \to \infty$

$$\tilde{\Sigma}_i \to \mathcal{C} \qquad \xi_i \to \xi_\infty \neq 0 \qquad \tilde{u}_i \to \tilde{u}_\infty$$

$$\partial_\tau \tilde{u}_\infty = \Delta_{\mathcal{C}} \tilde{u}_\infty + |A_{\mathcal{C}}|^2 \tilde{u}_\infty \qquad \text{on } \mathcal{C} \times (-\infty, 0]$$

$$\text{and } |\tilde{u}_\infty(\xi, \tau)| \leq \frac{1}{|\xi|^a} \quad \text{with equality at } (\xi_\infty, 0).$$

This contradicts the vanishing theorem for $\mathcal{C}$.

Case 3: $|x_i| \sim \sqrt{T - t_i}$

In this case, we can estimate the mean curvature directly from (Velázquez ’94).

$$\begin{aligned} M_i &= (1 + \Lambda_i |x_i|)^a |H_{\Sigma(t_i)}(x_i)| \\ &\sim \Lambda_i^a (T - t_i)^{a/2} \frac{1}{\sqrt{T - t_i}} (T - t_i)^{\lambda_k} \\ &= (T - t_i)^{\lambda_k (1 - \frac{a}{|\alpha| + 1}) - \frac{1}{2}} \\ &\ll 1 \end{aligned}$$

This contradicts the supposed blow-up of M_i. □

A Note on Constants

Let $n \geq 4$.

For any $a \in (|\alpha(n)|, |\alpha(n)| + 1)$, there always exist k large enough so that

$$\lambda_k \left(1 - \frac{a}{|\alpha| + 1}\right) - \frac{1}{2} \geq 0.$$

However, the “most generic” behavior $k = 2$ has

$$\lambda_2 \left(1 - \frac{a}{|\alpha| + 1}\right) - \frac{1}{2} < 0$$

for all $a \in (|\alpha|, |\alpha| + 1)$.

Indeed, (Guo-Šešum ’18) show that the mean curvature of $\Sigma_2^{2n-1}(t)$ blows up.

Conclusion

Bounds Outside the Parabolic Region

- (Ecker-Huisken '91) interior estimates imply uniform curvature estimates for $\Sigma_k^{2n-1}(t)$ when $|x| \geq r_0$.
- Sub-/Super-solutions control $\Sigma_k^{2n-1}(t)$ and H when $\sqrt{T-t} \leq |x| \leq r_0$.

These estimates plus the parabolic region bound show the Velázquez solution $\Sigma(t) = \Sigma_k^{2n-1}(t)$ has

$$\sup_{t \in [0,T)} \sup_{\Sigma(t)} |H| < \infty$$

when $n \geq 4$ and k is sufficiently large. $\square$

Applications to Weak Solutions

(Angenent-Daskalopoulos-Šešum '21) construct the smooth continuation of the Velázquez solutions for $t \in (T, T+\epsilon)$.

[Velázquez '94, S– '20, ADŠ '21]

$$\Downarrow$$

There's a MCF solution $\{\Sigma^7(t) \subset \mathbb{R}^8\}_{t\in[0,T+\epsilon)}$ such that

- $\Sigma(t)$ is smooth everywhere except $(\mathbf{0}, T) \in \mathbb{R}^8 \times [0, T+\epsilon)$,
- $\Sigma(t)$ has a type II singularity at $(\mathbf{0}, T)$, and
- $$\sup_{t\in[0,T+\epsilon), t\neq T} \sup_{\Sigma(t)} |H| < \infty.$$

Open Questions & Future Directions

- Can the Velázquez mean curvature flow solutions be compactified?
- (S– ’21) constructs (closed) Ricci flow solutions analogous to the Velázquez mean curvature flow solutions. Do these have uniformly bounded scalar curvature?
- Analogous constructions for other geometric flows.
- Removing $O(n) \times O(n)$-symmetry and obtaining singularities modeled on other minimal cones.
- What of dimensions $3 < N < 8$?

Thank you!