The Inverse Mean Curvature Flow and Minimal Surfaces

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BIRS New Directions in Geometric Flows

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Introduction

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Given a closed, oriented n-dimensional smooth manifold N, a one-parameter family of immersions $F : N \times [0, T) \to \mathbb{R}^{n+1}$ with outward normal ν and $H > 0$ is a **solution to Inverse Mean Curvature Flow (IMCF)** if

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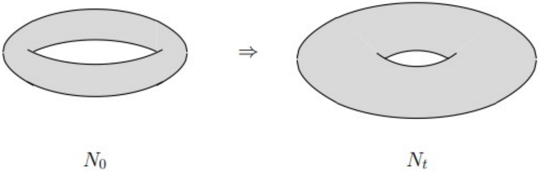
- Explicit solution: $N_0 = \mathbb{S}_R(x_0)$, then $N_t = \mathbb{S}_{r(t)}(x_0)$ for $r(t) = Re^{\frac{t}{n}}$.

Singularities of IMCF

- One example of singularity formation in IMCF is for a “thin” $(H > 0)$ torus $N_0 \subset \mathbb{R}^3$.



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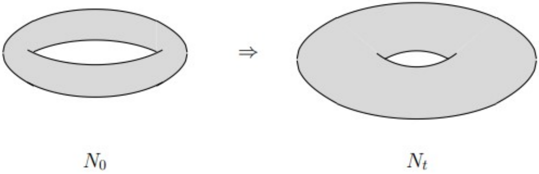

- One principal curvature is negative at the part of N_t closest to the axis of rotation.
- Since flow speed is bounded below, H must eventually reach 0 along this part, terminating the flow.

Characterizing the Thin Torus Singularity

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Theorem 1, The Limit of IMCF on a Torus

Let $N_0 = F_0(\mathbb{T}^2) \subset \mathbb{R}^3$ be an $H > 0$, rotationally symmetric embedded torus and $F : \mathbb{T}^2 \times [0, T_{\max}) \to \mathbb{R}^3$ the corresponding maximal solution to (1). Then $T_{\max} < +\infty$ and $\lim_{t \to T_{\max}} \max_{N_t} |A| \leq L < +\infty$.
Furthermore, there exists a subsequence of times $t_k \nearrow T_{\max}$ and corresponding diffeomorphisms $\alpha_k : \mathbb{T}^2 \to \mathbb{T}^2$ so that the maps $\widetilde{F}_{t_k} = F_{t_k} \circ \alpha_k : \mathbb{T}^2 \to \mathbb{R}^3$ converge in C^1 topology to an immersion $\widetilde{F}_{T_{\max}}$.

Rescaling the Singularity

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- Proof by contradiction: let $y_{\min}(t)$ be the distance from N_t to the axis of rotation and assume $\lim_{t \to T_{\max}} y_{\min}(t) = 0$.
- Define $\tilde{N}_t = \frac{1}{y_{\min}(t)} N_t$. Since $\max_{N_t} H \leq \max_{N_0} H$, we have

$$\lim_{t \to T_{\max}} \max_{\tilde{N}_t} H = 0. \tag{2}$$

- One expects some subset of $\tilde{N}_t$ to converge to a catenoid.

The Energy Method

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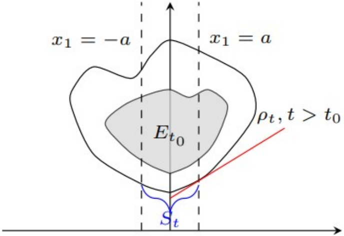

Figure: A cross section of the un-scaled surface N_t.

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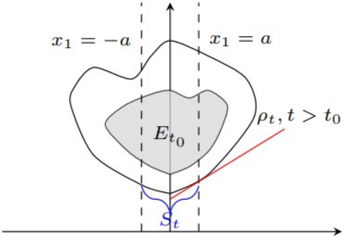

Figure: A cross section of the un-scaled surface N_t.

- $S_t \subset N_t$ between $y =$ const. planes has Gauss curvature K satisfying

$$\int_{S_t} |K| d\mu \leq 4\pi(1 - \epsilon) \tag{3}$$

for some $\epsilon > 0$.

The Contradiction

- Let $\tilde{S}_t = \frac{1}{y_{\min}(t)} S_t$, then

$$\int_{\tilde{S}_t} |K| d\mu \leq 4\pi(1-\epsilon). \tag{4}$$

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- On the other hand, generating graphs $y_t(x)$ of $\tilde{S}_t$ (or a subsequence) converge in $C^2_{\text{loc}}(\mathbb{R})$ to a catenary $y(x) = \frac{1}{a}\cosh(ax)$.
- $y(x)$ generates a catenoid C with $\int_C |K| d\mu = 4\pi$: this is a contradiction.

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- $y(x)$ generates a catenoid C with $\int_C |K| d\mu = 4\pi$: this is a contradiction.
- This implies $\sup_{N_t} |A| \leq C(N_0)$– the limit immersion is guaranteed by a compactness theorem from [Lan85].

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- Conjecture: $\sup_{N\times[0,T_{\max})}|A| \leq C(N_0)$ for any solution $\{N_t\}_{0\leq t<T_{\max}}$ of IMCF in $\mathbb{R}^3$.
- Remark: one can show

$$\sup_{t\in[0,T_{\max})} ||A||_{L^2(N_t)} \leq C(N_0) \tag{5}$$

for any immersed solution in $\mathbb{R}^3$. Does this energy concentrate?

Long-Time Existence in IMCF

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Theorem 2, [Ger90], Long-Time Existence in Star-Shaped IMCF

Let $N_0 \subset \mathbb{R}^{n+1}$ be an $H > 0$, strictly star-shaped hypersurface. Then for the corresponding maximal solution $\{N_t\}_{0 \leq t < T_{\max}}$ to IMCF, $T_{\max} = +\infty$ and N_t is strictly star-shaped for each $t \in [0, +\infty)$.

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- [Har20a] contains a similar result for rotationally symmetric spheres.

Theorem 3, Long-Time Existence in Rotationally Symmetric IMCF

Let $N_0 \subset \mathbb{R}^{n+1}$ be an $H > 0$, rotationally symmetric embedded sphere with principal curvature p of rotation satisfying

$$\frac{\max_{N_0} p}{\min_{N_0} p} < n^{\frac{n}{2(n-1)}}.$$

Then for the corresponding maximal solution $\{N_t\}_{0 \leq t < T_{+\max}}$, $T_{\max} = +\infty$ and N_t is a cylindrical graph (away from the axis of rotation) for each $t \in [0, +\infty)$.

The Number and Embeddedness of Area-Minimizers

- Given a Jordan curve $\gamma \subset \mathbb{R}^3$, how many stable minimal disks does it bound, and are they embedded?

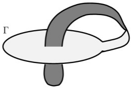

Figure: Source: [Cos12]

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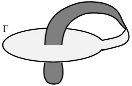

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Theorem 4, [MY82], Area-Minimizers in Mean-Convex Domains

Let $M \subset \mathbb{R}^3$ be a bounded domain with $\partial M \cong \mathbb{S}^2$ a C^2, $H > 0$ surface. For any Jordan curve $\gamma \subset \partial M$, γ bounds an immersed disk $D \subset M$ which minimizes area among all other immersed disks in M bounded by γ. Furthermore, this D is embedded.
Also, if γ is $C^{4,\alpha}$, then for any $k \in \mathbb{R}$ there are only finitely many stable minimal disks in M with areas less than k.

Minimal Disks and IMCF

- It is possible that γ bounds minimal disks in $\mathbb{R}^3$ which exit the domain M.
- In particular, solutions to Plateau's problem for this γ may not be embedded, and the finiteness property may not hold over $\mathbb{R}^3$.

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- From [Har20b], [Har20a], IMCF rules above possibilities out for certain M.

Theorem 5, Embeddedness and Finiteness of Area-Minimizers

Let $M \subset \mathbb{R}^3$, $\gamma \subset \partial M$ be as in the previous theorem. Suppose $N_0 = \partial M$ admits a long-time embedded solution $\{N_t\}_{0 \leq t < +\infty}$ to IMCF. Then all stable minimial disks bounded by γ lie in M.
In particular, the solution to Plateau's problem for γ is embedded, and if γ is $C^{4,\alpha}$ then it bounds only finitely many stable minimal disks with areas less than k.

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- The second part of the above result holds in particular for star-shaped or admissibly rotationally symmetric $H > 0$ domains.

The Comparison Principle

- If $\mathbb{R}^3 \setminus M$ is foliated by embedded, mean-convex closed surfaces, then for any immersed C^2 surface D with $\partial D \subset \partial M$ and $D \not\subset M$, $H(x) > 0$ for some $x \in D$.

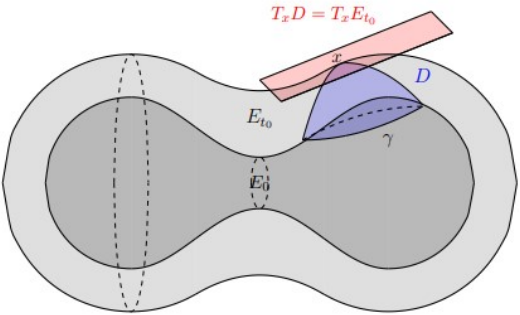

Figure: Let $\lambda_i, \tilde{\lambda}_i$ be the principal curvatures of D, ∂E_{t_0} at x respectively. Then $\lambda_i \geq \tilde{\lambda}_i$ and hence $H_D(x) > 0$.

Conditions for a Mean-Convex Foliation

- Caution: global solutions to IMCF need not remain embedded, and may also fail to foliate a region.
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Lemma 1, Foliations by IMCF

Let $\{N_t\}_{0\leq t<T}$ be a solution to IMCF. Then the N_t foliate the region $\cup_{0\leq t<T} N_t \subset \mathbb{R}^{n+1}$ if and only if N_t is embedded for each $t \in [0, T)$.

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- Proof relies on a "one-sided" avoidance principle for IMCF.

You have reached the time T_{max}.