Iterated monodromy groups of conservative polynomials

HOFSTRA UNIVERSITY

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Totally Disconnected Locally Compact Groups via Group Actions

Note

Most results are due to two groups of people, one involving Grigorchuk, Nekrashevych, and the speaker,
and another, involving a large group of undergraduate students, participating in Polymath Jr 2021 (NSF supported summer initiative, online collaboration open to students all over the world) and the speaker.

Name	Affiliation	Name	Affiliation
Ibrahim Abdelaziz	AU Cairo	Edisher Kipiani	St Andrews
Moaaz Alqady	AU Cairo	Juntao Lu	
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Alexander Cernei	CUNY	Nicholas Pandelakis	Carleton
Matthew Cohen	College NJ	Jaewoo Park	Michigan
John Connell*	South Florida	Micah Pietraho	Harvard
Sanjit Dandapanthula	UCLA	Soumyadip Sanyal	Duke/China
Gabriel Diraviam	Vanderbilt	Jeremy Sorkin	Wooster
Arthur DressenWall	Macalester	Jack Stade	Cambridge
Mrinmoyee Guha	Yale	Andy Wang	Princeton
Shalini Hewage	Colombo		

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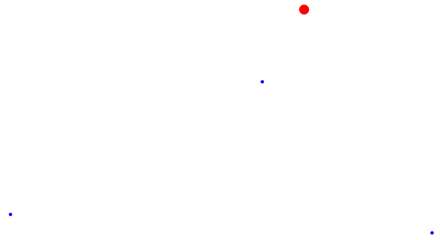
I will also mention one result with Garrido.

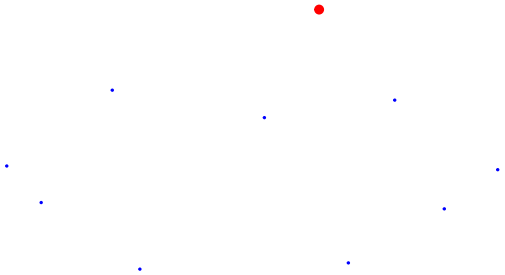
The backward iterates of a randomly chosen point, generation 0

Backward iterates of a random point $f(z) = \frac{3z - z^3}{2}$

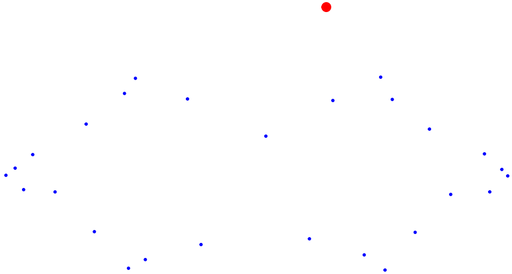
The backward iterates of a randomly chosen point, generation 1



The backward iterates of a randomly chosen point, generation 2

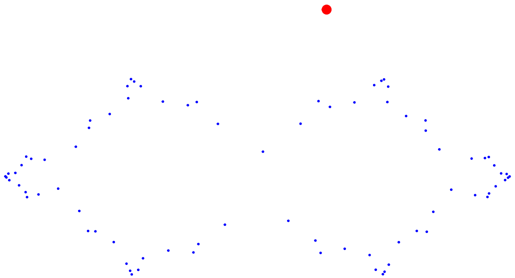


The backward iterates of a randomly chosen point, generation 3

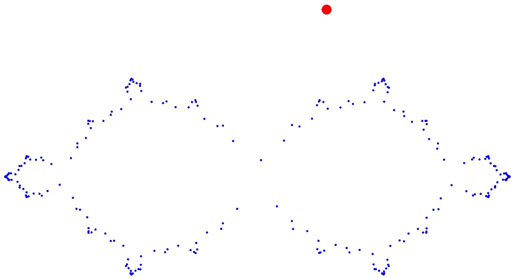


Backward iterates of a random point $f(z) = \frac{3z-z^3}{2}$

The backward iterates of a randomly chosen point, generation 4

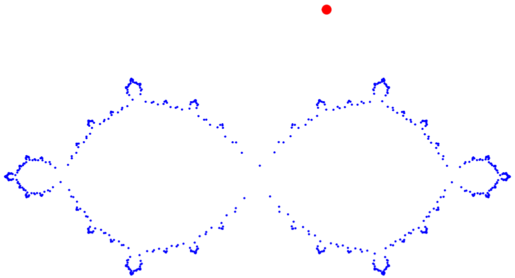


The backward iterates of a randomly chosen point, generation 5



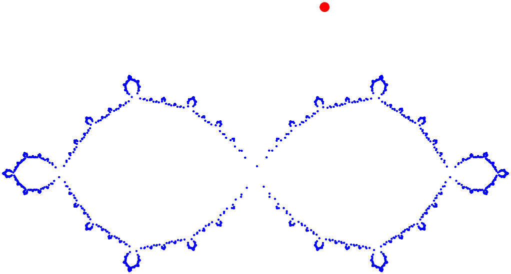
Backward iterates of a random point $f(z) = \frac{3z - z^3}{2}$

The backward iterates of a randomly chosen point, generation 6

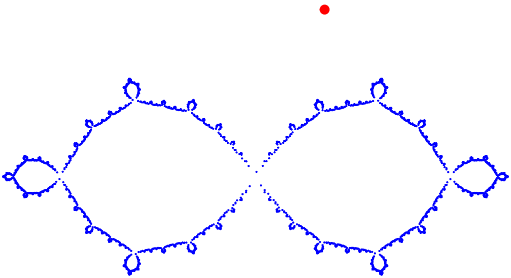


Backward iterates of a random point $f(z) = \frac{3z - z^3}{2}$

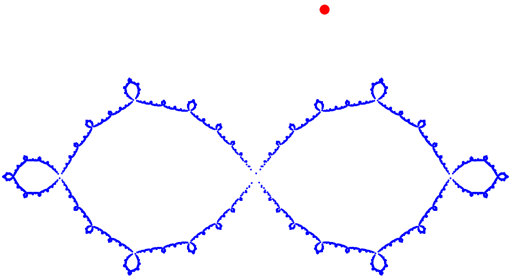
The backward iterates of a randomly chosen point, generation 7



The backward iterates of a randomly chosen point, generation 8

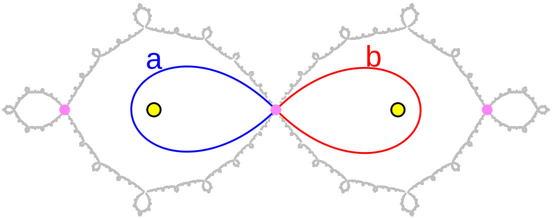


The backward iterates of a randomly chosen point, generation 9



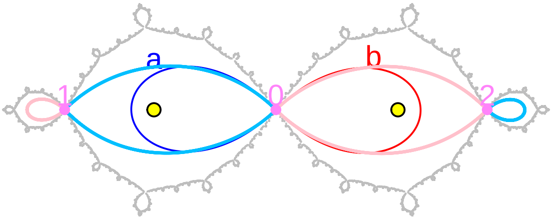
Backward iterates of two loops

$f(z) = \frac{3z - z^3}{2}$



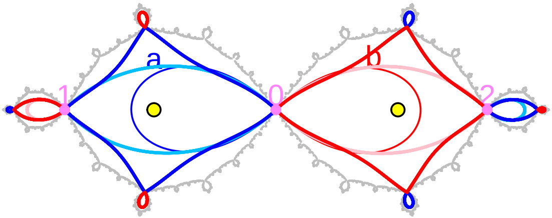
Backward iterates of two loops

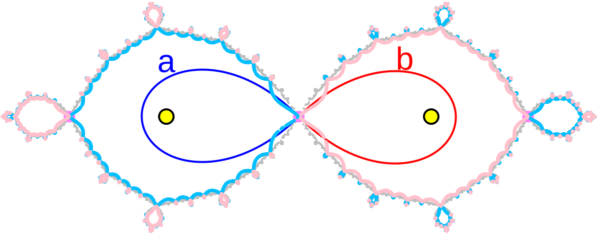
$f(z) = \frac{3z - z^3}{2}$

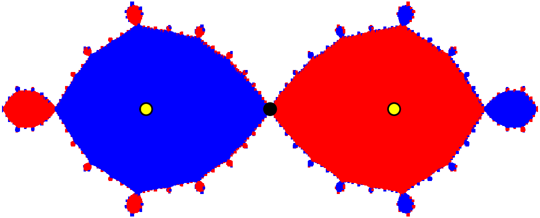


Backward iterates of two loops

$$f(z) = \frac{3z - z^3}{2}$$




a
b

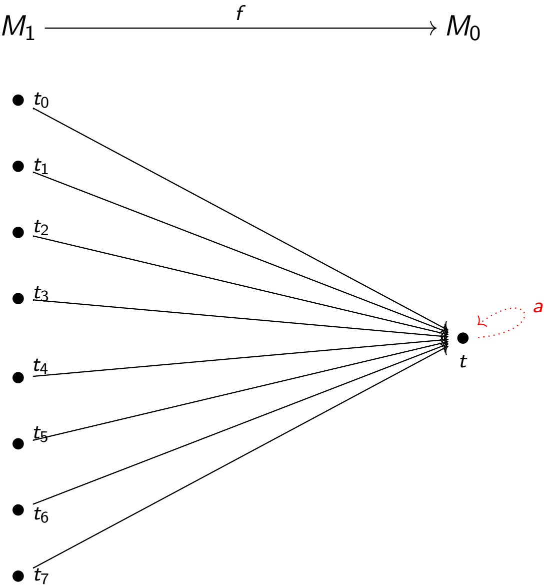


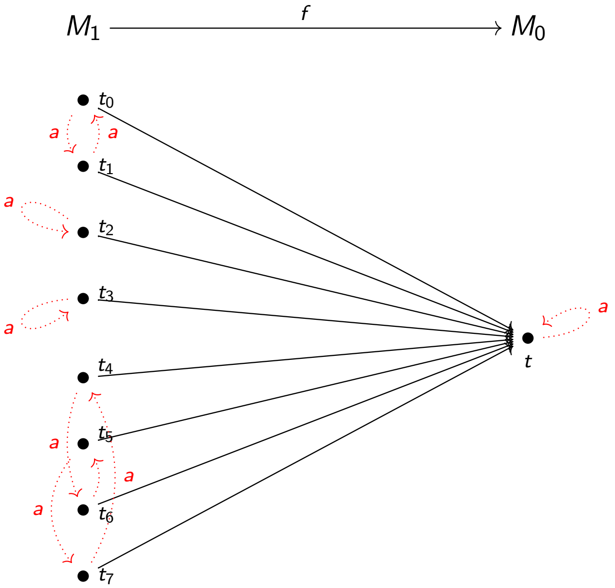
Monodromy group and iterated monodromy group

Let $f : M_1 \to M_0$ be d-to-1 covering map, t any point in M_0, and $t_0, t_1, \dots, t_{d-1}$ the d preimages of t under f.
Any loop a based at t lifts to d paths starting at $T = t_0, t_1, \dots, t_{d-1}$.
The loop a induces a (bijective) transformation $a : T \to T$ given by

$$a(t_x) = \text{ the end point of the lift of } a \text{ starting at } t_x$$

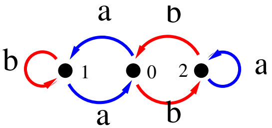
This is the **monodromy action** (monodromy transformation) of the loop a.
The **monodromy group** of f is the subgroup of $\mathrm{Sym}(T)$ consisting of the monodromy transformations induced by the fundamental group of M_0 based at t.

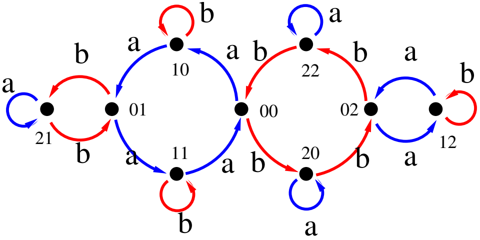

M1
f
M0
t0
t1
t2
t3
t4
t5
t6
t7
t
a



(01)(2)(3)(4657)

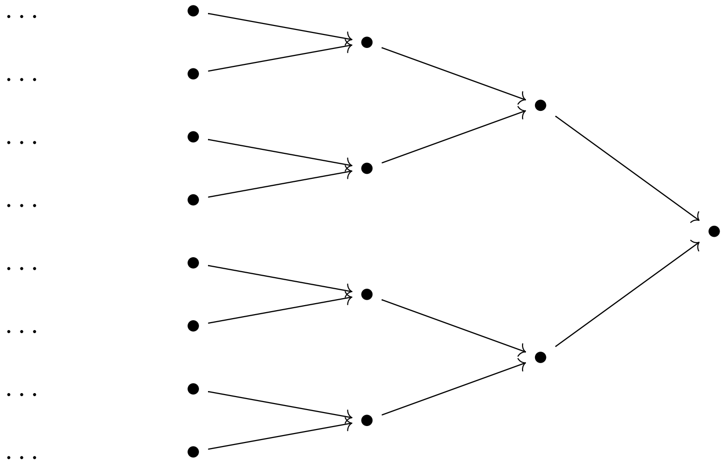
A 3-to-1 covering (take $t = 1$ and lift a^2, ab^2a)



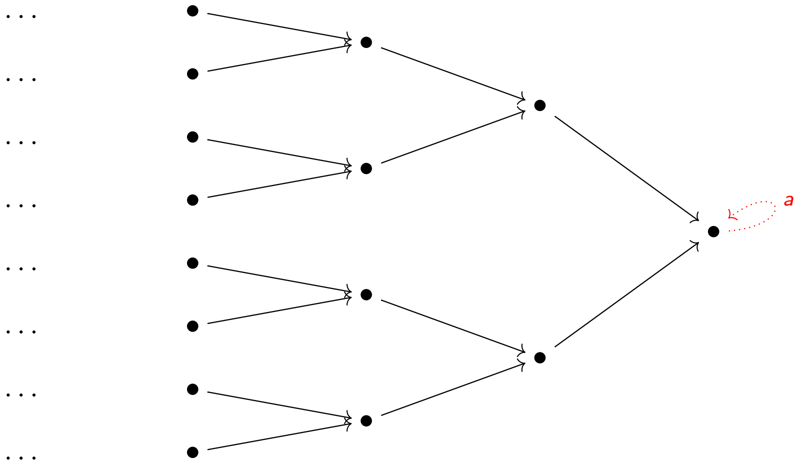


$$
\begin{array}{lll}
aa(10) = 11 & aa(11) = 10 & aa(12) = 12 \\
ab^2a(10) = 11 & ab^2a(11) = 12 & ab^2a(12) = 10
\end{array}
$$

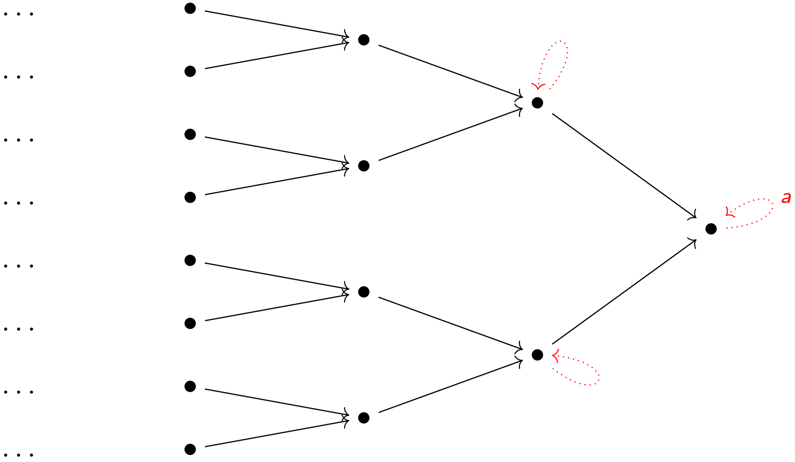
$$\cdots \xrightarrow{f} M_3 \xrightarrow{f} M_2 \xrightarrow{f} M_1 \xrightarrow{f} M_0$$



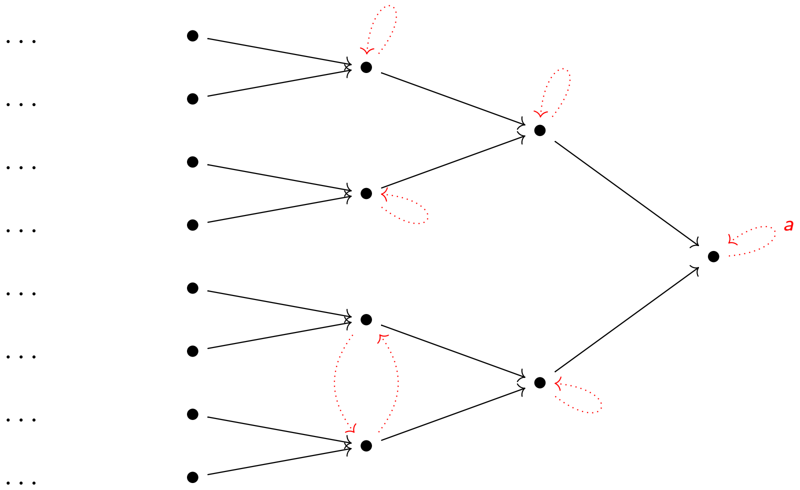
$$\cdots \xrightarrow{f} M_3 \xrightarrow{f} M_2 \xrightarrow{f} M_1 \xrightarrow{f} M_0$$



$$\cdots \xrightarrow{f} M_3 \xrightarrow{f} M_2 \xrightarrow{f} M_1 \xrightarrow{f} M_0$$



$$\cdots \xrightarrow{f} M_3 \xrightarrow{f} M_2 \xrightarrow{f} M_1 \xrightarrow{f} M_0$$



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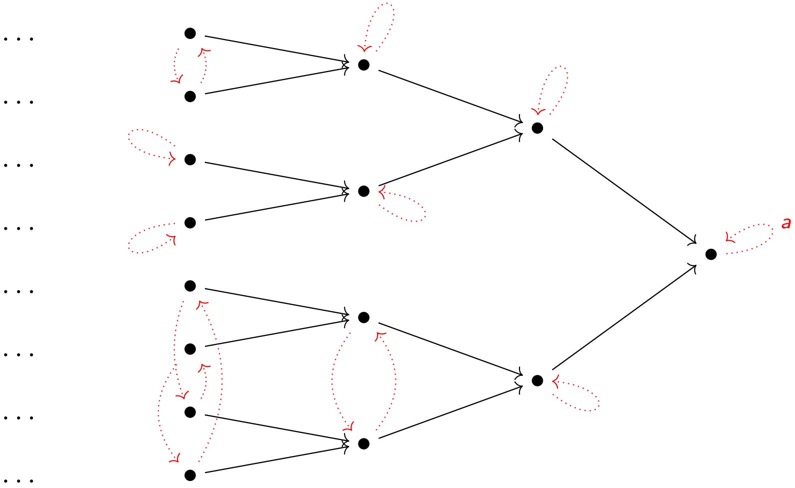


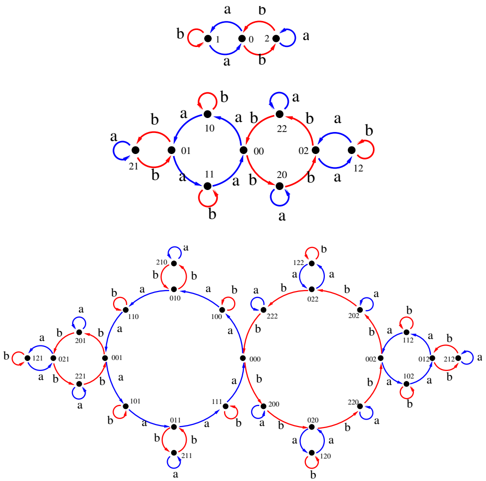
Definition of IMG

Definition (Nekrashevych)

Each loop based at t in M_0 induces an automorphism of the tree of pre-images of t. The group of all these tree automorphisms is called the iterated monodromy group of f.
Thus,

$$IMG(f) = \pi_1(M_0)/Ker(\text{action of } \pi_1 \text{ on the tree of preimages})$$

Lift twice (take $t = 1$ and lift ab^2a)



$$ab^2a : 100 \to 110 \to 121 \to 101 \to 111 \to 000 \to 120 \to 102 \to 112 \to 122$$

Post-critically finite polynomials

f = a post-critically finite polynomial of degree d

$f(z) = 2z^3 - 3z^2 + 1/2$

$f'(z) = 6z(z-1)$ so critical points 0 and 1 with orbits

$$\begin{tikzcd} 0 \bullet \arrow[r, bend left, "2"] & \bullet \tfrac{1}{2} \arrow[l, bend left] & 1 \bullet \arrow[r, bend left, "2"] & \bullet -\tfrac{1}{2} \arrow[loop above] \end{tikzcd}$$

$M_0 = \mathbb{C} - \mathrm{PCV}(f)$ (the post-critical value set removed)

$M_1 = f^{-1}(M_0) = \mathbb{C} - f^{-1}(\mathrm{PCV}(f))$ (the post-critical value set removed along with its preimages)

$M_0 = \mathbb{C} - \{1/2, \ -1/2, \ 0\}$ and $M_1 =$ $\mathbb{C} - \{1/2, \ -1/2, \ 0\} \cup \{\text{two preimages of } 0\} \cup \{\text{one preimage of } 1/2\}$

Then f restricted to

$$f : M_1 \to M_0$$

is a d-to-1 covering.

Moreover, $M_1 \subseteq M_0$ and we can keep taking preimages and lift loops forever.

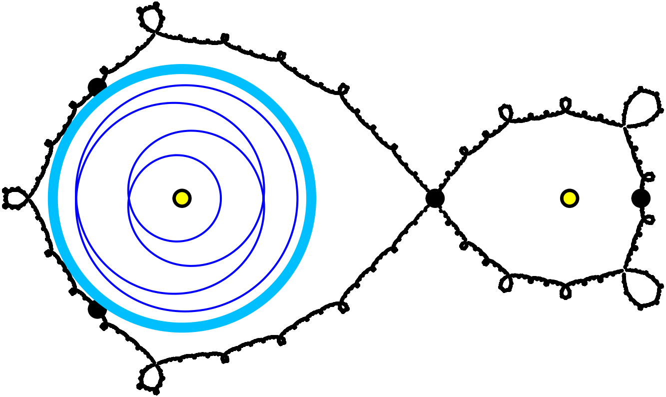
$$\dots \xrightarrow{f} M_4 \xrightarrow{f} \quad \xrightarrow{f} M_3 \xrightarrow{f} M_2 \xrightarrow{f} M_1 \xrightarrow{f} M_0$$

Conservative polynomials

Definition (Smale)

A polynomial is conservative if all of its critical points are fixed (in particular, such polynomials are post-critically finite).

$f_{4,3}(z) = 15z^4 - 24z^5 + 10z^6$: Behavior near 0



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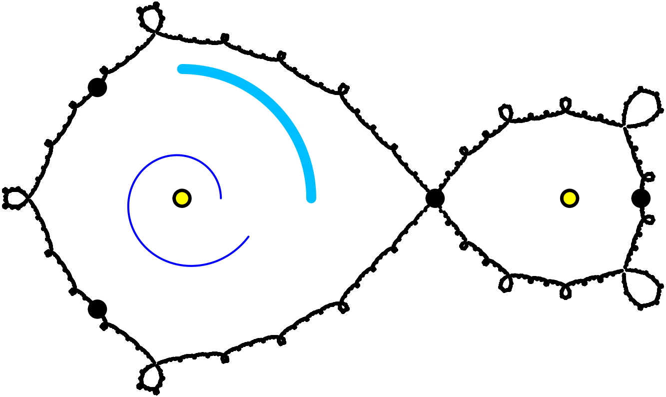

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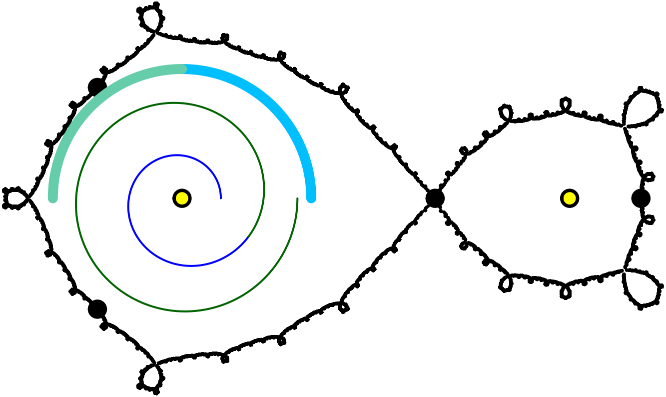
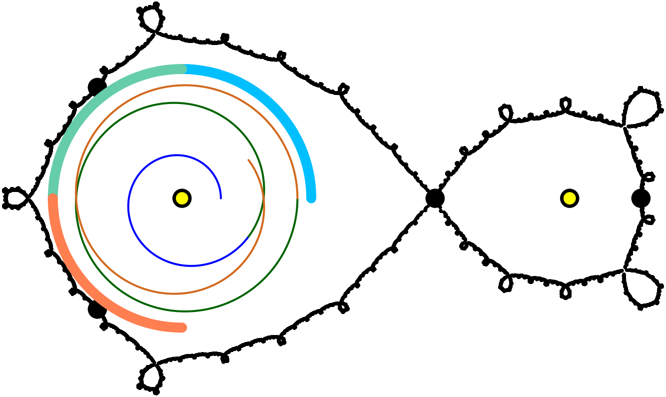
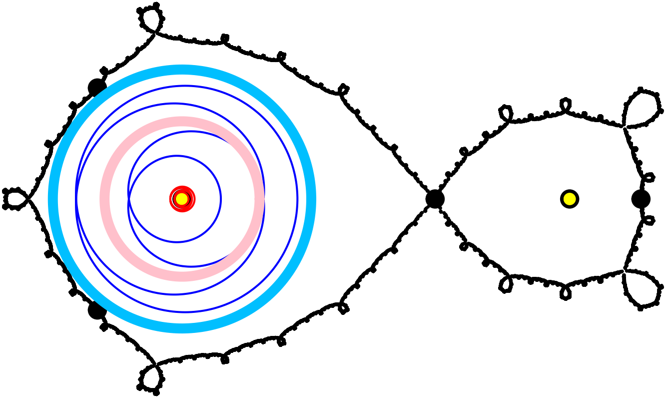


$f_{4,3}(z) = 15z^4 - 24z^5 + 10z^6$: Behavior near 0



Near 0, the term $15z^4$ in $15z^4 - 24z^5 + 10z^6$ dominates ...

... and everything looks more and more like $z \mapsto z^4$

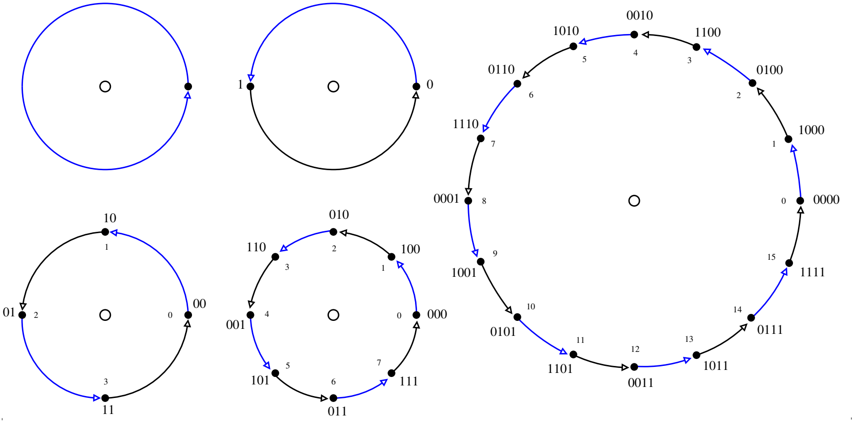


What about near 1?

We have

$$f(z)-1 = 5z^4-4z^5-1 = 20(z-1)^3+45(z-1)^4+36(z-1)^5+10(z-1)^6$$

Near 1, the term $20(z-1)^3$ dominates, things wrap around 1 thrice and behave like $z \mapsto z^3$



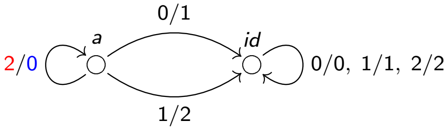
Applying f deletes the last digit.
The loop a acts by "adding 1" on the tree of preimages of $t = 1$.

$$a(0w) = 1w \qquad a(1w) = 0a(w)$$

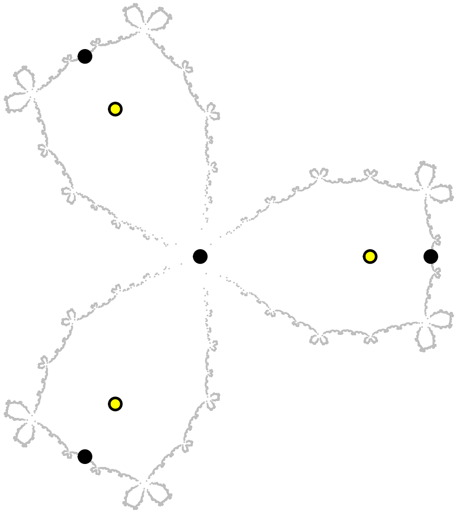
Ternary odometer (adding 1 ternary)

$$(012)$$

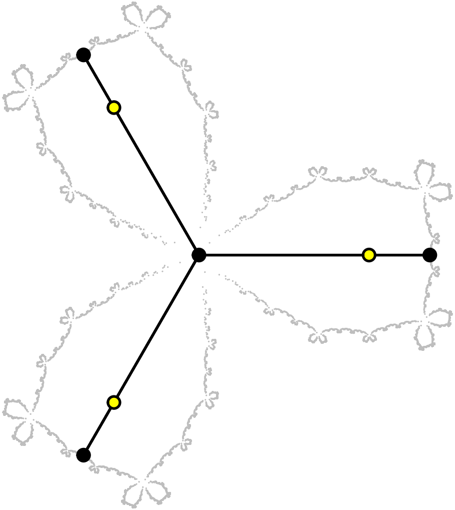
$$a(0w) = 1w \qquad a(1w) = 2a(w) \qquad a(2w) = 0a(w)$$



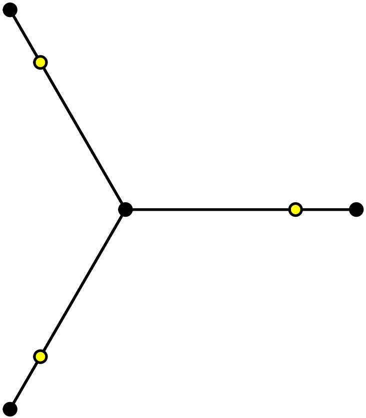
Parametrization by bi-colored trees



Parametrization by bi-colored trees



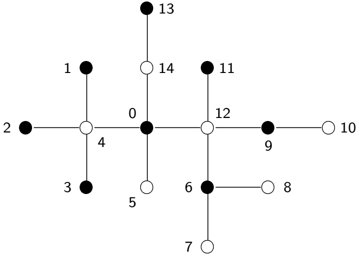
Parametrization by bi-colored trees

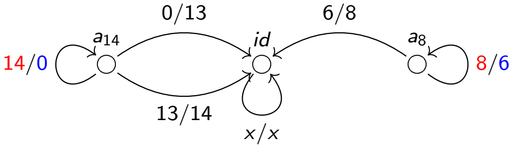


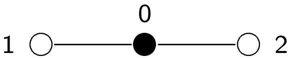
From bi-colored trees to finite transducers

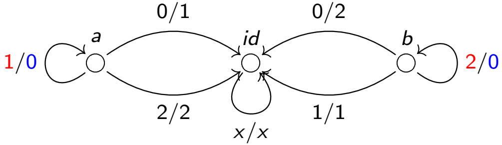
$$\alpha_4 = (0, 1, 2, 3, 4) \qquad \alpha_5 = (05) \qquad \alpha_7 = (6, 7) \qquad \alpha_8 = (6, 8)$$

$$\alpha_{10} = (9, 10) \qquad \alpha_{12} = (0, 6, 9, 11, 12) \qquad \alpha_{14} = (0, 13, 14)$$









$G = \langle a, b \rangle$ acts on words over $X = \{0, 1, 2\}$ as follows

$$\begin{array}{ll} a(0w) = 1w & b(0w) = 2w \\ a(1w) = 0aw & b(1w) = 1w \\ a(2w) = 2w & b(2w) = 0bw \end{array}$$

They act as binary odometers over two binary alphabets that share the 0 digit, but not not the nonzero digit.

Rooted ternary tree X^* $X = \{0, 1, 2\}$

X^* is the set of all words over the 3-letter alphabet $X = \{0, 1, 2\}$.

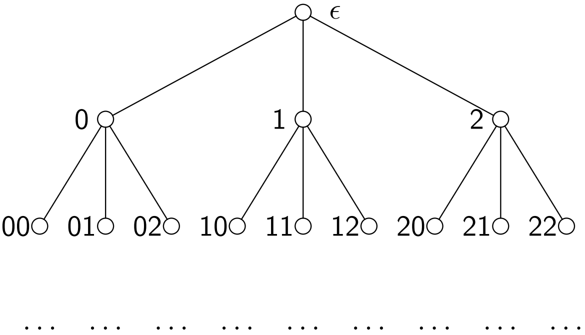
$X^* = \{\epsilon, 0, 1, 2, 00, 01, 02, 10, 11, 12, 20, 21, 22, \dots\} = X^0 \cup X^1 \cup X^2 \cup \dots$

$X^0 =$ level 0

$X^1 =$ level 1

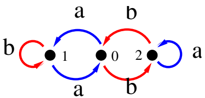
$X^2 =$ level 2

$\dots$

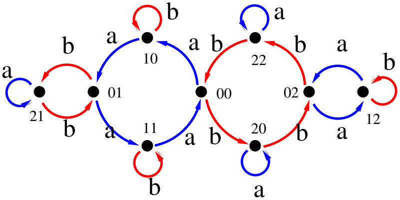


Schreier graph = graph of the action of a and b

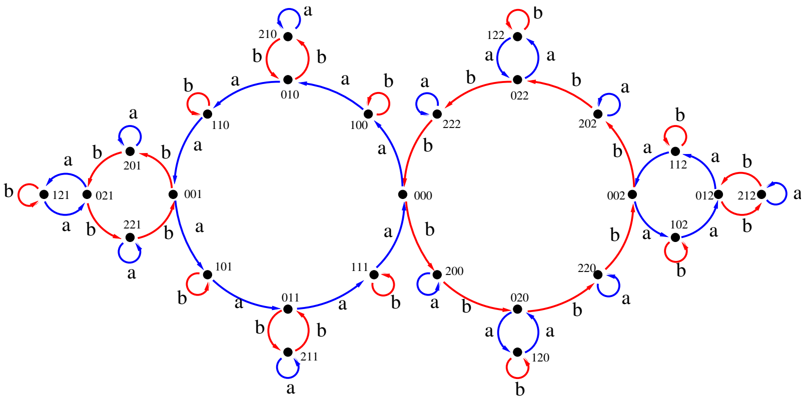
Level 1



Level 2



Level 3


210
010
110
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222
202
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121
021
001
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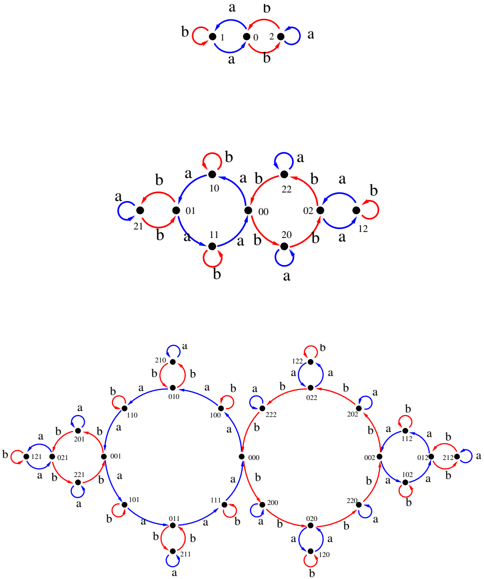
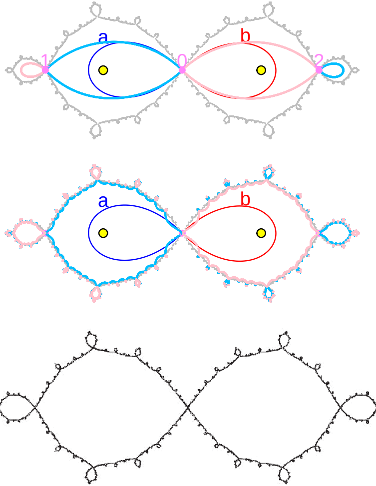
002
112
012
212
221
102
101
111
200
220
011
020
211
120
a
b

The sequence of Schreier graphs approximates the Julia set





The topological closure of $IMG(f)$ (the even case)

Theorem

Let f be a conservative polynomial and $G = \mathrm{IMG}(f)$.
If all critical points of f have even multiplicities, then

$$G = \mathrm{Alt}(d) \wr_X G = \mathrm{Alt}(d) \ltimes \underbrace{G \times G \times \cdots \times G}_{d}$$

and, consequently, G is a finitely generated, self-replicating, regular branch group, branching over itself.

Corollary

The group G is dense in the iterated permutational wreath product

$$\mathrm{Alt}(d) \wr_X (\mathrm{Alt}(d) \wr_X (\mathrm{Alt}(d) \wr \dots$$

In other words, the closure of G consists of all tree automorphisms whose local permutation at every vertex is even.

The topological closure of $IMG(f)$ (the odd case)

Theorem

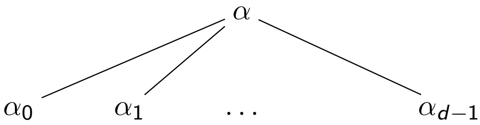
Let f have at least one odd multiplicity. Denote by E the subgroup of index 2 in G consisting of the elements whose top permutation is even. Then G is a finitely generated, self-replicating, regular branch group, branching over E. Moreover,

$$G \leq_2 \mathrm{Sym}(d) \wr_X G = \mathrm{Sym} \ltimes \underbrace{G \times G \times \cdots \times G}_{d}$$

consisting of the elements $\alpha(g_0, \dots, g_{d-1})$ for which the product of the top permutations in $g_0, \dots, g_{d-1}$ has the same parity as α.

In other words, the closure consists of all tree automorphisms for which the parity of the local permutation at every vertex agrees with the total parity of the local permutations of its d children.

On closures of branch groups



parity of α = sum of the parities of $\alpha_0, \alpha_1, \dots, \alpha_{d-1}$

Theorem (Garrido-Š)

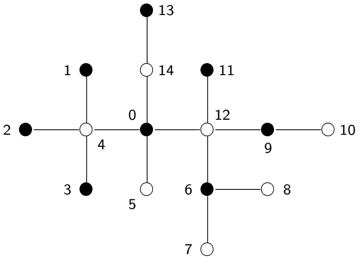
Let G be a self-similar, self-replicating, regular branch group of tree automorphisms in $\mathrm{Aut}(X^*)$*. Then its closure in* $\mathrm{Aut}(X^*)$ *is finitely constrained (a group tree-shift of finite type).*

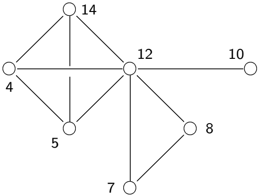
The semigroup generated by the odometers

Theorem

Let f be a conservative polynomial, $G = \mathrm{IMG}(f)$, with the standard generating set S coming from the bi-colored tree. The semigroup generated by S is the free partially commutative semigroup on S (called trace in CS), with the commuting (called independence in CS) relations

$$a_x a_y = a_y a_x \iff a_x \text{ and } a_y \text{ share a digit}$$





Theorem

The map

$$a \mapsto a^2 \qquad b \mapsto b^2$$

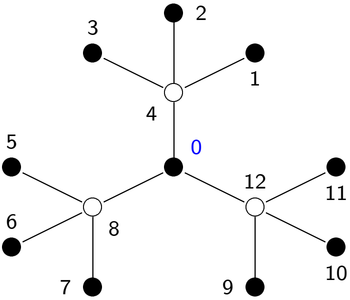
extends to an injective homomorphism $\phi : G \to G$.
A presentation for $G(2;2)$ *is given by*

$$\langle a, b \mid R \cup \phi(R) \cup \phi^2(R) \cup \dots \}$$

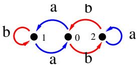
where

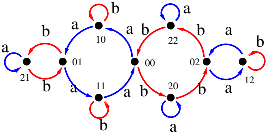
$$\begin{aligned} R = \{ & (aba^{-1}b^{-1})^3, \\ & aba^{-1}b^{-1}abab^{-1}aba^{-1}b^{-1}a^{-2}, \\ & bab^{-1}a^{-1}baba^{-1}bab^{-1}a^{-1}b^{-2}, \\ & ab^{-1}a^{-1}baba^{-1}b^{-1}(a^{-1}b^{-1}a^2ba^{-1}bab^{-1}a^{-1})^{-1} \} \end{aligned}$$

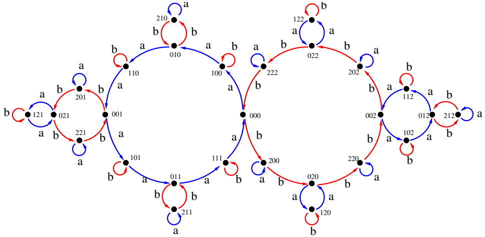
$k = \#$ of odometers $= \#$ of critical points $=$ the rank of π_1
$r =$ the odometer arity $=$ local degree near critical pts $=$ multiplicity $+1$
$d =$ the degree of the polynomial $= \#$ fixed points $= \#$ vertices $=$ $k(r-1)+1$
The k odometers share the digit 0 but none of the other digits.

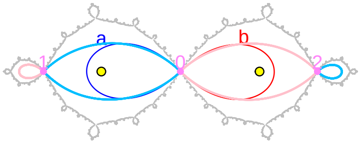


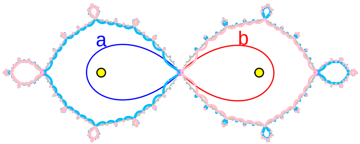
The spectrum of $G(2,2)$

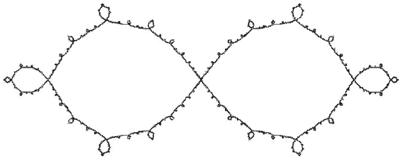












Spectrum of G(2;2)

$$P_{n+1}(x) = (x-2)^{3^n} \cdot P_n(x^2 - 2x - 4)$$

Denote $q(x) = x^2 - 2x - 4$.

$$\begin{aligned}
P_0(x) &= x - 4 \\
P_1(x) &= (x-2)(q(x) - 4) \\
P_2(x) &= (x-2)^3(q(x) - 2)(q(q(x)) - 4) \\
P_3(x) &= (x-2)^{3^2}(q(x) - 2)^3(q(q(x)) - 2)(q(q(q(x))) - 4)
\end{aligned}$$

So, the spectrum on level n is

$$q^{-n}(4) \cup \bigcup_{i=0}^{n-1} q^{-i}(2)$$

Spectrum of G(r;k)

Fix k and set $k_0 = k - 1$.
Let p_r and q_r, be the polynomials recursively defined by

$$p_0(x) = 1, \qquad q_0(x) = 2k,$$
$$p_1(x) = x - 2k_0, \qquad q_1(x) = x,$$
$$p_r(x) = (x - 2k_0) \cdot p_{r-1}(x) - p_{r-2}(x), \qquad q_r(x) = (x - 2k_0) \cdot q_{r-1}(x) - q_{r-2}(x).$$

When $k = 2$,

$$p_0(x) = 1 \qquad q_0(x) = 4$$
$$p_1(x) = x - 2 \qquad q_1(x) = x$$
$$p_2(x) = x^2 - 4x + 3 \qquad q_2(x) = x^2 - 2x - 4$$
$$p_3(x) = x^3 - 6x^2 + 10x - 4 \qquad q_3(x) = x^3 - 4x^2 - x + 8$$
$$\ldots$$

Spectrum of G(r;k)

Theorem

The characteristic polynomial of the adjacency matrix of the nth level Schreier graph of $G(r;k)$ is given recursively by

$$P_{n+1}(x) = (p_{r-1}(x))^{(k-1)d^n} P_n(q_r(x))$$

The spectrum of the corresponding infinite Schreier graph (coming from the action on right-infinite words) is a set of isolated points together with its closure, which is a Cantor set (and is the Julia set of q_r)

The sequence of scaled Laplacians $r^n L_n$ on level n converges to a Laplacian on the Julia set of f.

Theorem

The spectrum of the adjacency matrix of the nth level Schreier graph of the monoid $M(r;k)$ is given by backward iterations of the polynomial

$$q_r(x) = x(x-k_0)^{r-1}$$