Group actions and power maps

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- **Group action:** Suppose G and X are two lc groups. Action of G on X (by automorphisms) is a continuous map $\phi\colon G \times X \to X$ such that for each $g \in G$, $\phi_g(x) = \phi(g, x)$ is an automorphism of X and $g \mapsto \phi_g$ is a homomorphism of G.

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(3) If H is a closed normal subgroup of G, then G acts on H by inner autormorphisms (restricted to H).

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X is a normal subgroup of $G \ltimes X$

Tdlc groups and scale function

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Theorem [Wi-94]

- $s(\alpha) = 1 = s(\alpha^{-1})$ if and only if α fixes a compact open subgroup U, that is, $\alpha(U) = U$.
- $s(\alpha^n) = s(\alpha)^n$.
- the scale s is continuous on G.

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Aim: Structural conditions related to power map having dense image or surjective.
We first recall the following

Theorem [Wi-94]

If G is a tdlc group and $g_n^{k_n} \to g$ a and $k_n \to \infty$ as $n \to \infty$, then $s(g) = 1$.

If $P_k(G)$ is dense in G, then for $g \in G$, using the continuity of P_k, we can find a sequence (g_n) in G such that $g_n^{k^n} \to g$.

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Proposition [MaR-20]

If G as above acts on a tdlc group X, then each $g \in G$, fixes a compact open subgroup of X.

linear actions

An automorphism α of a lc group X is called distal if e is not a limit point of $\{\alpha^n(x) \mid n \in \mathbb{Z}\}$ for any $x \in X \setminus \{e\}$.

The following gives structural condition for linear actions by distal maps.

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The following gives structural condition for linear actions by distal maps.

Theorem [CoG-74]

Let G be a subgroup of $GL(V)$. Then the following are equivalent:

- each $\alpha \in G$ is distal on V.
- eigenvalues of each $\alpha \in G$ are of absolute value one.
- there is a G-invariant flag of subspaces $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = V$ such that all orbits of G in V_i / V_{i-1} are bounded.

Assumption

- V a finite-dimensional vector space over a non-Arichmedean local field $\mathbb{F}$.

G is a tdlc group for which

- $P_K\colon G \to G$ is dense and
- $\rho\colon G \to GL(V)$ is a continuous representation of G. Using the representation, we consider the action of G on V and obtain

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Theorem [MaR-20]

There is a flag of subspaces with associated unipotent group U and a compact group L such that $\rho(G) \subset LU$ and the flag is L-invariant.

Groups associated to a flag

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- U is called unipotent group associated to the flag and H is the invaraint group of the flag. U is a subgroup of H.
- $\beta \in H$ fixes V_i implies U is a normal subgroup of H.
- The action of H on U is linear in the sense that there is a H-invariant central series $U_0 \subset U_1 \subset \cdots \subset U_n = U$ such that U_0 is trivial and each U_i/U_{i-1} is a vector space and corresponding H-action on U_i/U_{i-1} is linear.

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Theorem [DaM-17]

Let L be a compact totally disconnected group and N be a nilpotent lc group. Suppose L acts on N and the action is linear over a field $\mathbb{F}$. If P_k is surjective on L and k is co-prime to the characteristic of $\mathbb{F}$, then P_k is surjective on $L \ltimes N$.

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Contd.,

We obtain the following

Theorem [MaR-20]

Let $\mathbb{F}$ be a non-Archimedean local field and G be a group with a linear representation $\rho\colon G \to \mathrm{GL}(d, \mathbb{F})$. Suppose that P_k is dense in G for some $k > 1$. Then we have the following:

- There exists a compact group $L \subset \mathrm{GL}(d, \mathbb{F})$ and a split unipotent algebraic group $U \subset \mathrm{GL}(d, \mathbb{F})$ normalized by L such that $L \cap U$ is trivial, $\rho(G) \subset LU$ and $\rho(G)U$ is dense in LU. Moreover, P_k is surjective on $LU/U \simeq L$.
- If k is co-prime to the characteristic of $\mathbb{F}$, then P_k is surjective on LU.
- If the characteristic p of $\mathbb{F}$ divides k, then $\rho(G)$ is finite.

Residual characteristic

In the case of characteristic of $\mathbb{F}$ not dividing k, considering the residual characteristic and obtain the following:

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Theorem [MaR-19]

- If the residual characteristic p of $\mathbb{F}$ divides k, then L is finite, that is $\rho(G)$ is contained in a finite extension of a split unipotent algebraic group U and P_k is dense in $\rho(G) \cap U$.
- If the residual characteristic p of $\mathbb{F}$ divides k and the characteristic of $\mathbb{F}$ is zero (resp., positive), then $\overline{\rho(G)}$ is a finite extension of a split unipotent algebraic group (resp., finite).

Corollaries

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- Any compactly generated group linear group over $\mathbb{F}$ for which P_k is dense is compact.

Assume that $\mathbb{E}$ is a global field and H is a linear group over $\mathbb{E}$ such that P_k is surjective on H for some $k > 1$.

- If the characteristic of $\mathbb{E}$ is 0, then H contains an unipotent normal subgroup of finite index (see [Ch-09] for related results).
- If the characteristic of $\mathbb{E}$ is $p > 0$, then H is locally finite, that is any finitely generated subgroup of H is finite.
- If the characteristic p of $\mathbb{E}$ divides k, then H is finite.
- If P_k is surjective on H for all $k \geq 1$, then either H is a unipotent group or H is trivial depending on characteristic of $\mathbb{E}$ is 0 or positive. .

Algebraic groups

Assume that $\mathbb{F}$ is a non-Archimedean local field and G is an algebraic group defined over $\mathbb{F}$: p-adic algebraic group case is considered in [Ch-09].

Theorem [Mar-20]

Let $R_{us,\mathbb{F}}(G)$ be the $\mathbb{F}$-split unipotent radical of G. Suppose that the characteristic of $\mathbb{F}$ does not divide k. Then the following are equivalent:

(a) P_k is dense in $G(\mathbb{F})$;

(b) $G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F})$ is compact and P_k is surjective on $G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F})$;

(c) P_k is surjective in $G(\mathbb{F})$.

Suppose the residual characteristic of $\mathbb{F}$ divides k. Then density of P_k on $G(\mathbb{F})$ implies that $G(\mathbb{F})$ is a finite extension of a split unipotent group. In addition if characteristic of $\mathbb{F}$ is positive, then $G(\mathbb{F})$ is finite.

Algebraic groups contd.,

Theorem [MaR-19]

- If P_k is surjective on $G(\mathbb{F})$ and H is an algebraic subgroup of G defined over $\mathbb{F}$, then P_k is surjective on $H(\mathbb{F})$.
- If H is a closed (not necessarily algebraic) normal subgroup in $G(\mathbb{F})$ and P_k is dense in H as well as in $G(\mathbb{F})/H$, then P_k is surjective on $G(\mathbb{F})$
- Suppose P_k is surjective on $G(\mathbb{F})$ for all $k \in \mathbb{N}$. Then $G(\mathbb{F})$ is unipotent. In addition if characteristic of $\mathbb{F}$ is positive, then $G(\mathbb{F}) = \{e\}$.

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Thanks for your attention!!!