Combining the Runge approximation and the Whitney embedding theorem in hybrid imaging

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Internal data in quantitative hybrid imaging problems

- Hybrid conductivity imaging [Widlak, Scherzer, 2012]

$$\begin{cases} -\mathrm{div}(a\,\nabla u_i) = 0 & \text{in } \Omega, \\ u_i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

$u_i(x)$ or $a(x)\,\nabla u_i(x)$ or $a(x)\,|\nabla u_i|^2(x)$ $\overset{?}{\longrightarrow}$ a

- Quantitative thermoacoustic tomography [Bal et al., 2011, Ammari et al., 2013]

$$\begin{cases} \Delta u_i + (\omega^2 + \mathrm{i}\omega\sigma)\,u_i = 0 & \text{in } \Omega, \\ u_i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

$\sigma(x)\,|u_i|^2(x)$ $\overset{?}{\longrightarrow}$ σ

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$$\begin{cases} \mathrm{curl} E^i = \mathrm{i}\omega H^i & \text{in } \Omega, \\ \mathrm{curl} H^i = -\mathrm{i}(\omega\varepsilon + \mathrm{i}\sigma)E^i & \text{in } \Omega, \\ E^i \times \nu = \varphi_i \times \nu & \text{on } \partial\Omega. \end{cases}$$

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Non-vanishing gradients and Jacobians

- Consider for simplicity the hybrid conductivity problem with internal data ∇u and unknown a:

$$\begin{cases} -\mathrm{div}(a\,\nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

- With 1 measurement:

$$\nabla a \cdot \nabla u = -a\Delta u \quad \Longrightarrow \quad \nabla(\log a)\cdot\nabla u = -\Delta u$$

This equation may be solved in a if a is known on $\partial\Omega$ and if

$$\nabla u(x) \neq 0, \qquad x \in \Omega.$$

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Main question

Is it possible to find suitable illuminations φ_i so that the corresponding solutions u_i satisfy certain non-zero constraints, such as a non-vanishing Jacobian

$$\det \begin{bmatrix} \nabla u_1(x) & \cdots & \nabla u_d(x) \end{bmatrix} \neq 0\,?$$

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1. The Radó-Kneser-Choquet theorem
2. Runge approximation & Whitney embedding
3. The multi-frequency method

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Theorem (Alessandrini Magnanini 1994, Bauman et al. 2000, Alessandrini Nesi 2015)

Let $\Omega \subseteq \mathbb{R}^2$ *be bounded and convex and* $a \in C^{0,\alpha}(\overline{\Omega}; \mathbb{R}^{2\times 2})$ *be uniformly elliptic. Let* $u_i \in H^1(\Omega)$ *solve*

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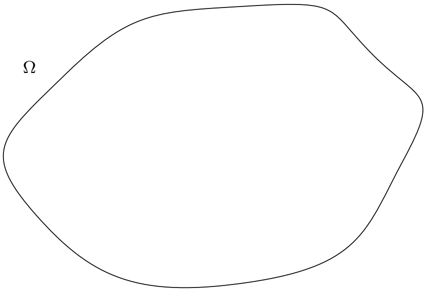

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- Thus, $\alpha\nabla u_1(x_0) + \beta\nabla u_2(x_0) = 0$
- Set $v(x) = \alpha u_1(x) + \beta u_2(x)$:
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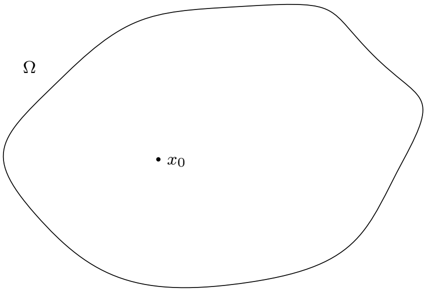

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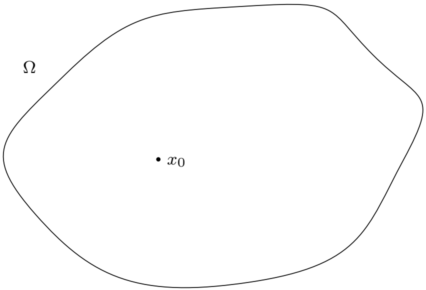

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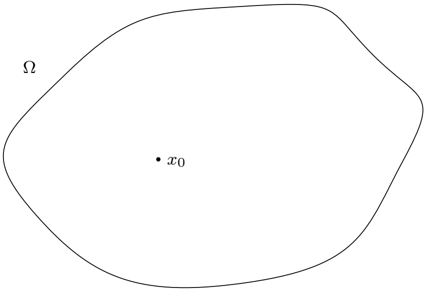

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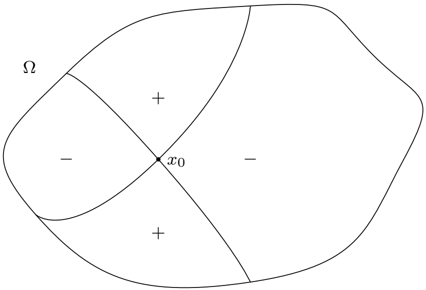

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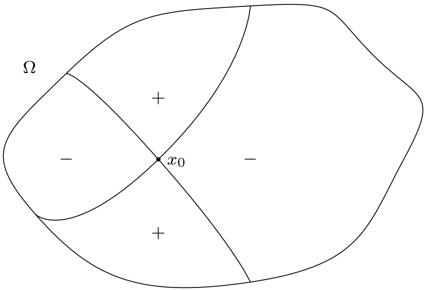

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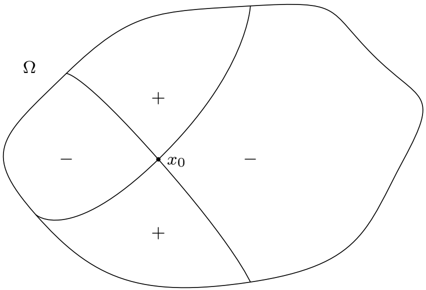

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The failure in 3D and for other elliptic PDEs

- In three dimensions, the above result fails. Counterexamples by Laugesen 1996, Briane et al 2004 and Capdeboscq 2015: it is not possible to find $(\varphi^1, \varphi^2, \varphi^3)$ independently of a so that

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Critical points in 3D

What about critical points: can we find φ independently of a so that

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Theorem (GSA, Bal, Di Cristo, ARMA 2017)

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz domain. Take $\varphi \in C(\partial X) \cap H^{\frac{1}{2}}(\partial X)$. There exists a (nonempty open set of) $a \in C^\infty(\overline{X})$ such that the solution $u \in H^1(X)$ to

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has a critical point in Ω, namely $\nabla u(x) = 0$ for some $x \in \Omega$.

Can be extended to deal with:

- multiple boundary values;
- multiple critical points (located in arbitrarily small balls);
- and Neumann boundary conditions.

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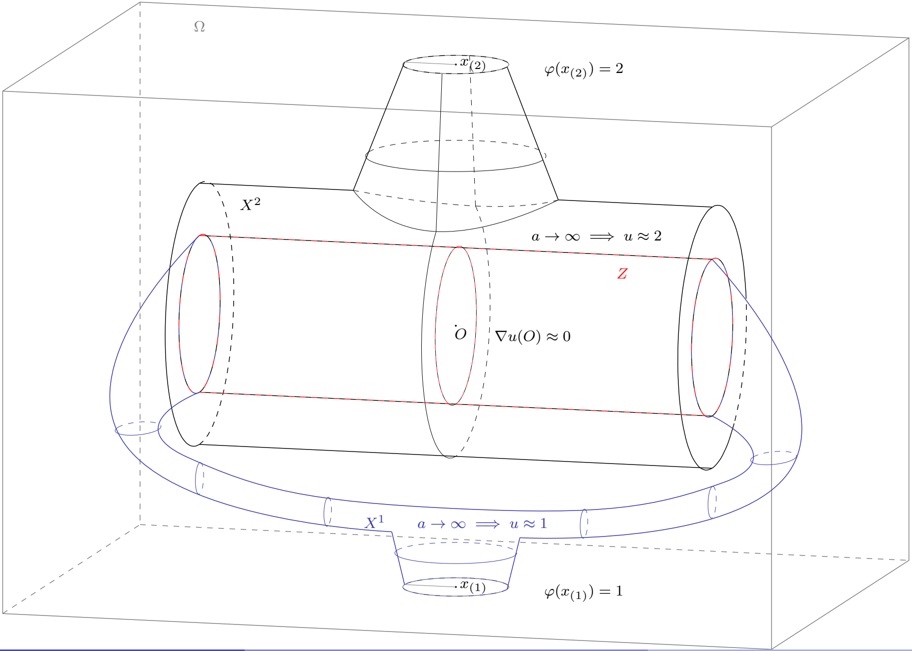
Ω
$x_{(2)}$
$\varphi(x_{(2)}) = 2$
X^2
$a \to \infty \implies u \approx 2$
Z
O
$\nabla u(O) \approx 0$
X^1
$a \to \infty \implies u \approx 1$
$x_{(1)}$
$\varphi(x_{(1)}) = 1$

Alternative approaches

- Complex geometrical optics solutions [Sylvester and Uhlmann, 1987]
  - $u^{(t)}(x) = e^{tx_m}(\cos(tx_l) + \mathrm{i}\sin(tx_l))(1+\psi_t), \quad t \gg 1.$
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Alternative approaches

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Outline of the talk

The model problem

- Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ be a smooth bounded domain. Consider the elliptic PDE

$$Lu := -\mathrm{div}(a\nabla u) + b \cdot \nabla u + cu = 0 \quad \text{in } \Omega,$$

with a, b and c smooth enough so that $u \in C^{1,\alpha}$ and the unique continuation property (UCP) holds

- No restrictions on dimension or on the PDE
- Consider, for simplicity, the non-vanishing Jacobian constraint: look for φ_i such that

$$\det \begin{bmatrix} \nabla u_1 & \cdots & \nabla u_d \end{bmatrix} (x) \neq 0$$

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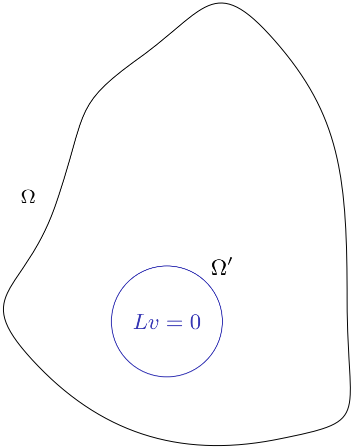
Main tool: the Runge Approximation [Lax 1956]



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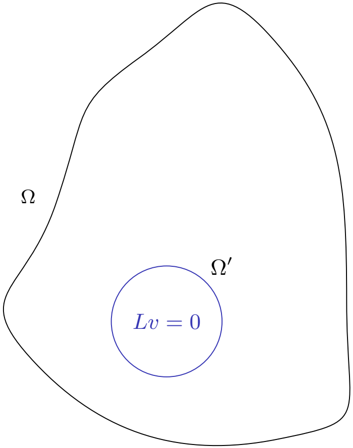

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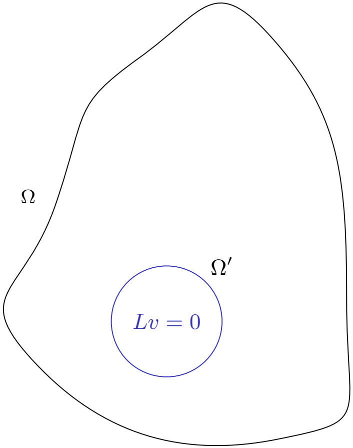

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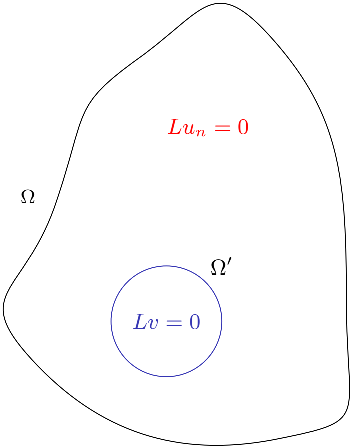

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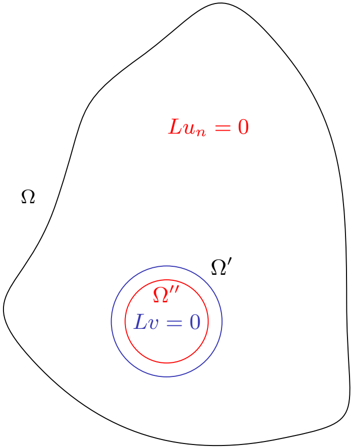

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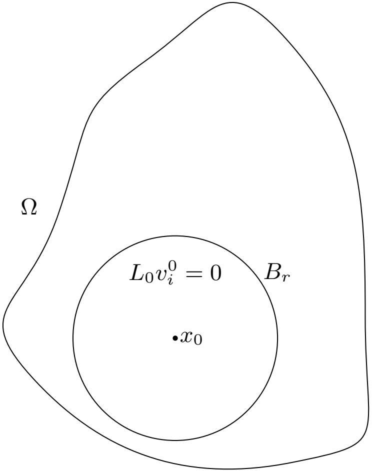

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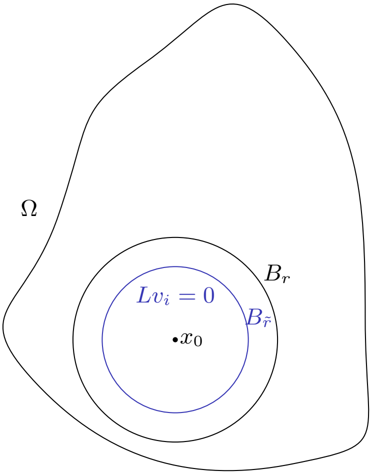

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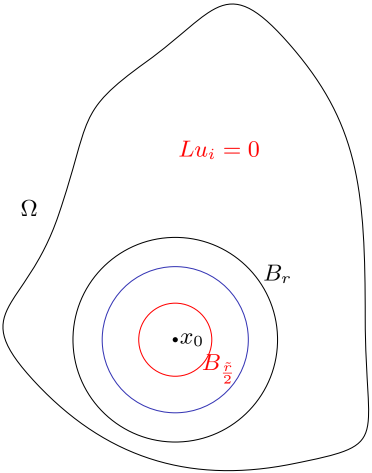

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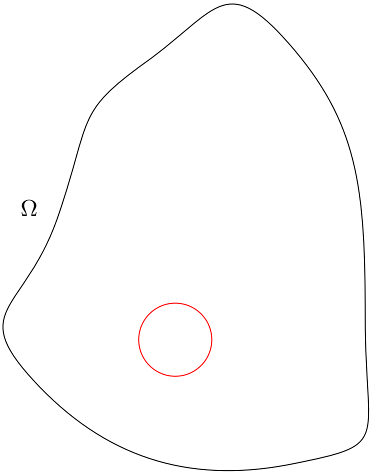

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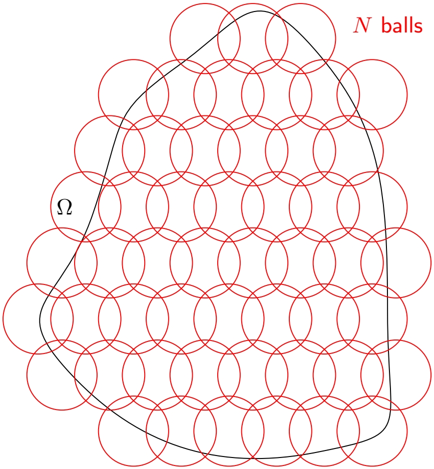

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Two main issues

- You need a large number of measurements to satisfy the constraint

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Whitney projection argument

Lemma (Greene and Wu 1975)

Take $k > 2d$ (possibly large). Let $u_1, \dots, u_k$ be solutions to $Lu_i = 0$ in Ω such that

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In other words: we can almost always reduce the number of solutions (until $2d$) and keep the constraint. In particular, arbitrarily small weights a can be used.

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Theorem (GSA and Capdeboscq 2019)

The set of $2d$ solutions $u_1, \dots, u_{2d}$ to $Lu_i = 0$ in Ω such that

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is open and dense in the set of $2d$ solutions to $Lu_i = 0$ in Ω.

Proof.

Open. The rank is stable under small perturbations of u_i.

Dense. Take $\tilde{u}_1, \dots, \tilde{u}_{2d}$ solutions to $L\tilde{u}_i = 0$. By Runge, we have a large number of solutions so that

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$$\operatorname{rank}\begin{bmatrix}\nabla \tilde{u}_1 & \cdots & \nabla \tilde{u}_{2d} & \nabla u_1 & \cdots & \nabla u_{Nd}\end{bmatrix}(x) = d, \qquad x \in \overline{\Omega}.$$

Apply Whitney reduction lemma with small weights a multiple (Nd) times, until you reach $2d$ solutions.

Remarks on the result

- As a corollary, the set of $2d$ boundary conditions whose solutions satisfy the constraint everywhere is open and dense.
- The approach is very general, and works with many other constraints, like

$$\begin{array}{ll} |u_1|(x) > 0 \text{ (nodal set)} & d+1 \text{ solutions} \\ \left|\det\begin{bmatrix}\nabla u_1 & \cdots & \nabla u_d\end{bmatrix}\right|(x) > 0 \text{ (Jacobian)} & 2d \text{ solutions} \\ \left|\det\begin{bmatrix}u_1 & \cdots & u_{d+1} \\ \nabla u_1 & \cdots & \nabla u_{d+1}\end{bmatrix}\right|(x) > 0 \text{ (``augmented'' Jacobian)} & 2d+1 \text{ solutions}\end{array}$$

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Outline of the talk

The Helmholtz equation

- We now consider the Helmholtz equation

$$\begin{cases} \Delta u_\omega^i + (\omega^2 \varepsilon + \mathrm{i}\omega\sigma)\, u_\omega^i = 0 & \text{in } \Omega, \\ u_\omega^i = \varphi_i & \text{on } \partial\Omega. \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, $\varepsilon, \sigma \in L^\infty(\Omega)$, $\sigma, \varepsilon \leq \Lambda$, $\varepsilon \geq \Lambda^{-1}$.

- We are interested in the constraints:
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Multi-Frequency Approach: main result

$K^{(n)}$: uniform partition of $\mathcal{A} = [K_{min}, K_{max}]$ with n points

Theorem (GSA, IP 2013 & CPDE 2015)

There exist $C > 0$ and $n \in \mathbb{N}^*$ depending only on Ω, Λ and $\mathcal{A}$ such that the following is true. Take

$$\varphi_1 = 1, \qquad \varphi_2 = x_1, \qquad \dots \qquad \varphi_{d+1} = x_d.$$

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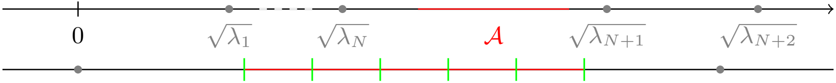

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As an example, let us consider the 1D case with $\varepsilon = 1$ and $\sigma = 0$.

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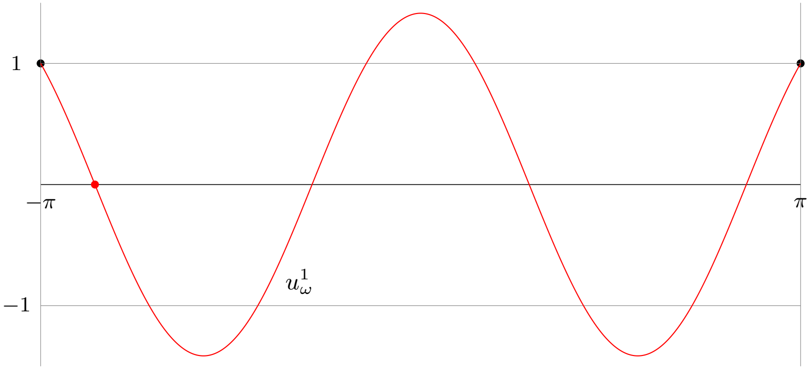

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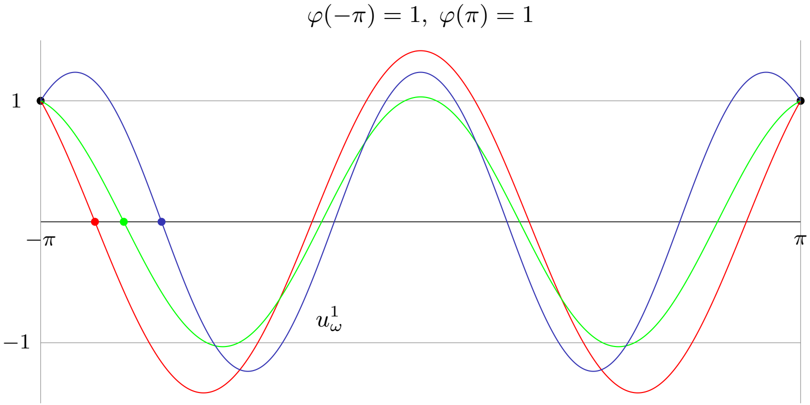

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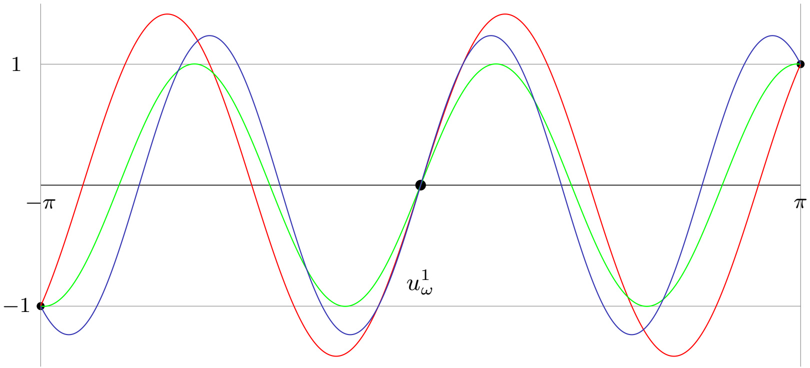
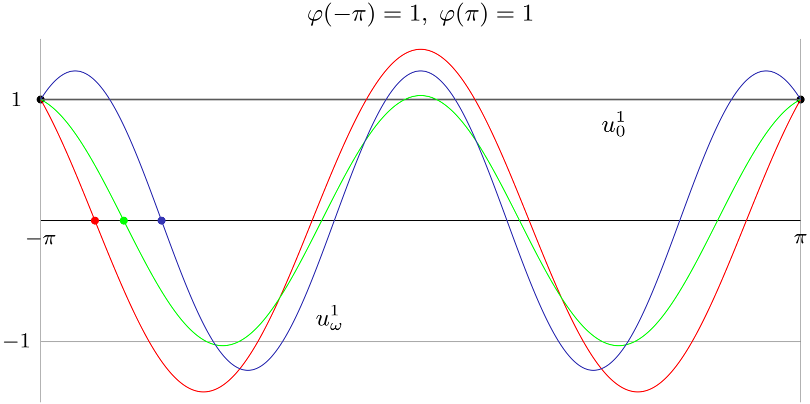


Multi-Frequency Approach: $\omega = 0$

1. $\left|u_0^1(x)\right| > 0$ everywhere for $\omega = 0$ $\implies$ the zeros "move"



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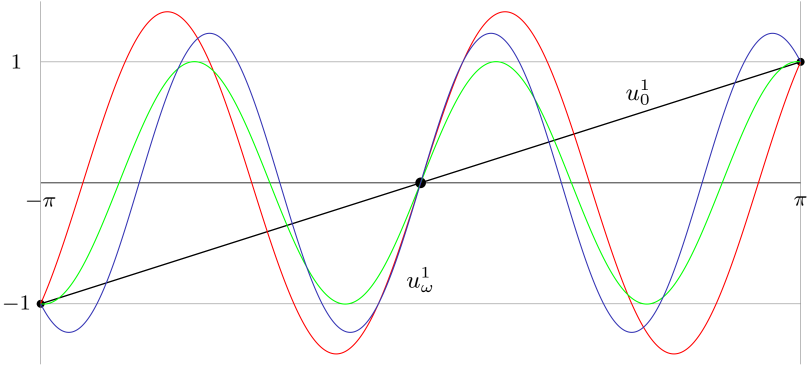

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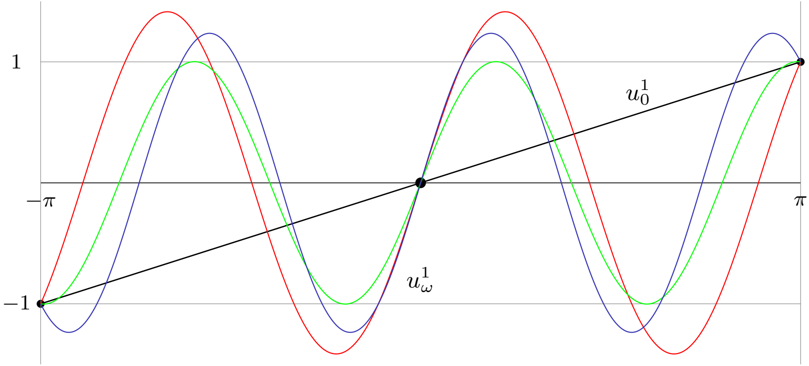

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Some related works

- Maxwell's equations (GSA, JDE 2015)
- Ammari et al. (2016) have successfully adapted this method to

$$\mathrm{div}((\omega\varepsilon + \mathrm{i}\sigma)\nabla u^i_\omega) = 0.$$

- In 2D, everything works with $a \in C^{0,\alpha}(\Omega; \mathbb{R}^{2\times 2})$ and

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  by using the absence of critical points for the conductivity equation.
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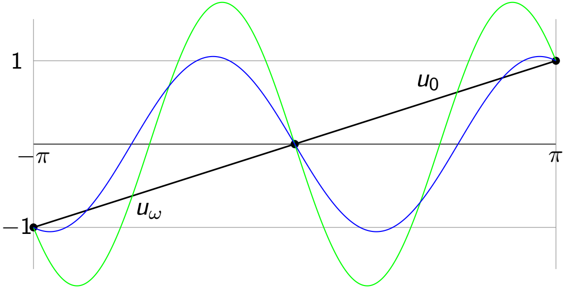

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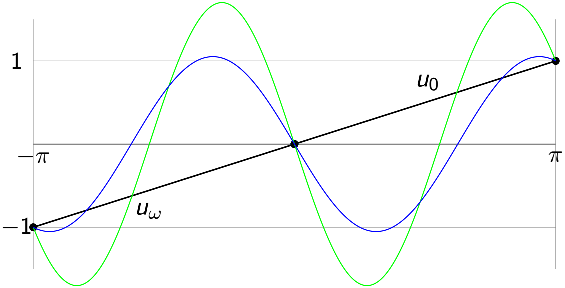

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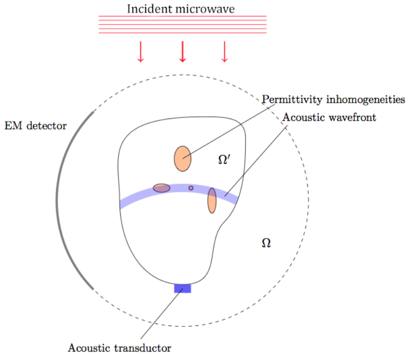
Acousto-electromagnetic tomography (Ammari et al., 2012)



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$$\psi_\omega = |u_\omega|^2 \nabla \varepsilon$$

- Linearised problem:
$$D\psi_\omega[\varepsilon](\rho) \mapsto \rho$$

In order to have well-posedness of the linearised inverse problem we need
$$\|D\psi_\omega[\varepsilon](\rho)\| \geq C \|\rho\|, \qquad \rho \in H^1(\Omega),$$

or equivalently $\ker D\psi_\omega[\varepsilon] = \{0\}$.

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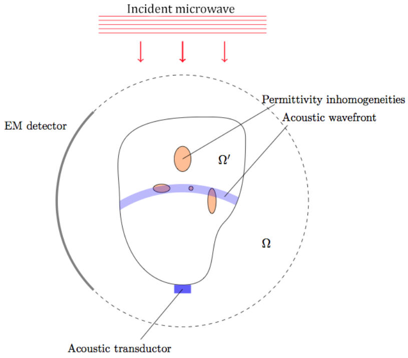

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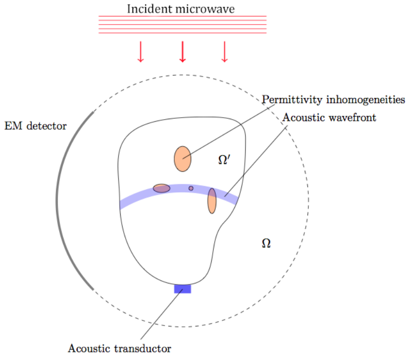

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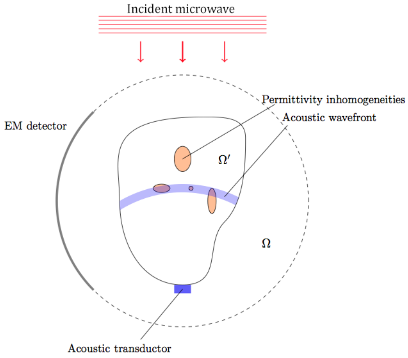

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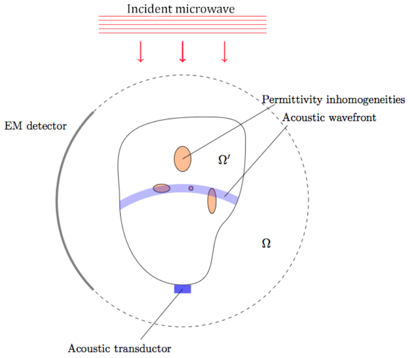

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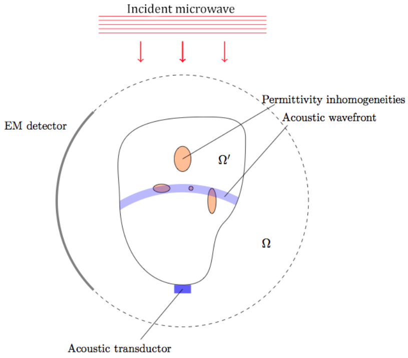

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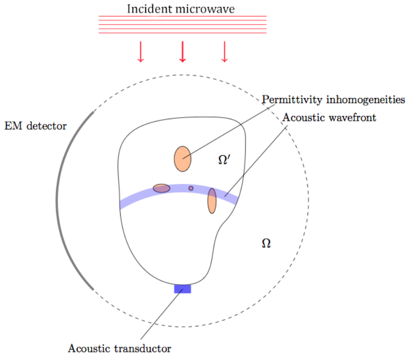
$$\sum_{\omega \in K} \|D\psi_\omega[\varepsilon](\rho)\| \geq C_K \|\rho\|, \qquad \rho \in H^1(\Omega),$$

or equivalently $\cap_{\omega \in K} \ker D\psi_\omega[\varepsilon] = \{0\}$.

Theorem (Alberti, Ammari, Ruan, 2014) This holds true with a priori determined frequencies K and stability constant C_K.

Acousto-electromagnetic tomography (Ammari et al., 2012)



- Model
$$\begin{cases} \Delta u_\omega + \omega^2 \varepsilon u_\omega = 0 \text{ in } \Omega, \\ \frac{\partial u_\omega}{\partial \nu} - i\omega u_\omega = \varphi \text{ on } \partial\Omega. \end{cases}$$
- Internal data:
$$\psi_\omega = |u_\omega|^2 \nabla \varepsilon$$
- Linearised problem:
$$D\psi_\omega[\varepsilon](\rho) \mapsto \rho$$

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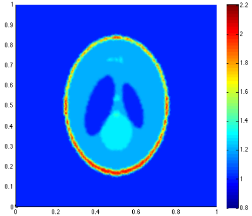
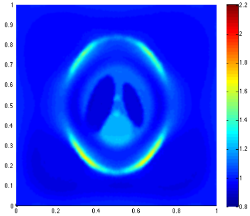
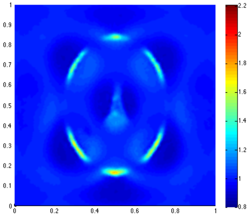
Numerical experiments

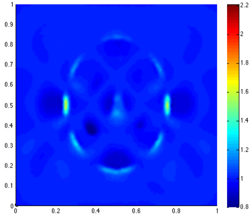




(a) $K = \{10\}$

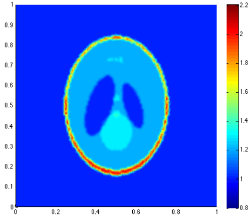


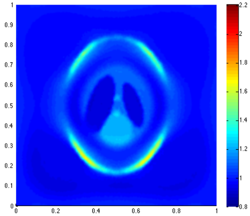
(b) $K = \{15\}$



(c) $K = \{20\}$

Numerical experiments





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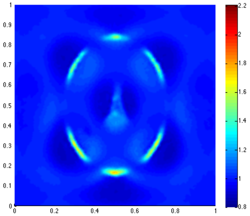

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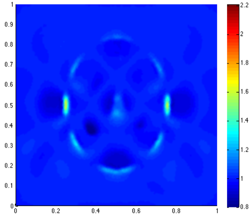

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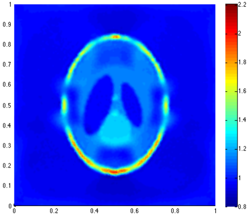

(d) $K = \{10, 15, 20\}$

Conclusions

- The inversion in quantitative hybrid imaging often requires the solutions to the direct problem to satisfy certain non-zero constraints.
- It is in general difficult to enforce these constraints a priori (independently of the unknown coefficients), but certain techniques are available:
  - The Radó-Kneser-Choquet theorem and its generalizations (only in 2D, not for Helmholtz)
  - CGO solutions
  - Runge & Whitney
  - The multi-frequency approach
- Future prospectives for Runge & Whitney:
  - complex coefficients
  - other PDEs (Maxwell, elasticity, etc.)
  - move from open and dense to high probability (or 1) with random boundary conditions

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Summer School on Applied Harmonic Analysis and Machine Learning

Genoa, September 9-13, 2019

Home Outline Schedule Info Registration

[~] *Three minicourses on Signal Analysis and Big Data*

School speakers:

Rima Alaifari (ETH Zurich)
Gabriel Peyré (École Normale Supérieure, Paris)
José Luis Romero (University of Vienna, Austrian Academy of Sciences)

Workshop speakers:

Massimo Fornasier (Technical University of Munich)
Anders Hansen (University of Cambridge)

Organizers:

Giovanni S. Alberti
Filippo De Mari
Ernesto De Vito
Lorenzo Rosasco
Matteo Santacesaria
Silvia Villa

Sponsors:

DIMA

Università di Genova

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kick-off event

July 1, 2019 | aula magna - via Balbi 5, Genova

9.30 am	Registration
9.30 am	Welcome addresses
10.00 am	Lorenzo Rosasco | *UniGe*
10.30 am	Nicolò Cesa-Bianchi | *UniMi*
11.10 am	Coffee Break
11.40 am	Yair Weiss | *The Hebrew University of Jerusalem*
12.20 pm	Tomaso Poggio | *MIT*
1.00 pm	Lunch buffet
2.30 pm	Presentation of the research units