The impact of conditional stability estimates on variational regularization and the distinguished case of oversmoothing penalties

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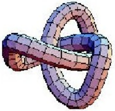

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The talk presents joint work with:

Daniel Gerth and Christopher Hofmann (Chemnitz)

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Frank Werner (Göttingen)

Papers which are relevant for the talk:

▷ F. NATTERER: Error bounds for Tikhonov regularization in Hilbert scales. *Applicable Anal.* **18** (1984), 29–37.

▷ B. HOFMANN, O. SCHERZER: Factors influencing the ill-posedness of nonlinear problems. *Inverse Problems* **10** (1994), 1277–1297.

▷ J. CHENG AND M. YAMAMOTO: On new strategy for a priori choice of regularizing parameters in Tikhonov's regularization. *Inverse Problems* **31** (2000), L31–L38.

▷ H. EGGER, B. HOFMANN: Tikhonov regularization in Hilbert scales under conditional stability assumptions. *Inverse Problems* **34** (2018), 115015.

▷ J. FLEMMING: *Quadratic Inverse Problems and Sparsity Promoting Regularization – Two Subjects, Some Links Between Them, and an Application in Laser Optics*. *Birkhäuser*, Basel 2018.

▷ F. WEIDLING, B. SPRUNG, AND T. HOHAGE: Optimal convergence rates for Tikhonov regularization in Besov spaces. arXiv:1803.11019, 2018.

▷ B. HOFMANN, P. MATHÉ: Tikhonov regularization with oversmoothing penalty for non-linear ill-posed problems in Hilbert scales. *Inverse Problems* **34** (2018), 015007.

▷ F. WERNER, B. HOFMANN: Convergence analysis of (statistical) inverse problems under conditional stability estimates. arXiv:1905.09765v1, 2019.

Outline

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Let X and Y denote infinite dimensional **Hilbert spaces**, equipped with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$.

We consider the (possibly non-linear) operator equation

$$F(x) = y \qquad (x \in \mathcal{D}(F) \subseteq X,\ y \in Y) \qquad (*)$$

as a model of an **inverse problem**

with forward operator $F : \mathcal{D}(F) \subset X \to Y$ and domain $\mathcal{D}(F)$.

Let $x^\dagger \in \mathcal{D}(F)$ denote the uniquely determined solution to $(*)$.

The **goal** is to find **stable** approximations to $x^\dagger$ with good properties based on **noisy data** $y^\delta \in X$ such that

$$\|y - y^\delta\|_Y \le \delta,$$

with noise level $\delta > 0$.

Since equation $(*)$ is the model of an **inverse problem**, the **forward operator** F is in general **'smoothing'**. Hence, a least squares approach

$$\|F(x) - y^\delta\|_Y^2 \to \min, \quad \text{subject to} \quad x \in \mathcal{D}(F),$$

is mostly not successful, even if $x^\dagger$ is the unique solution to $(*)$. Precisely, the stable approximate solution of $(*)$ requires some kind of **regularization**.

We exploit closed balls $\mathcal{B}_r^Z(\bar{z}) := \{z \in Z : \|z - \bar{z}\|_Z \le r\}$ and recall an ill-posedness concept adapted to nonlinear problems:

Definition ▷ H./SCHERZER IP 1994

The equation $(*)$ is called **locally well-posed** at the solution point $x^\dagger \in \mathcal{D}(F)$ if there is a ball $\mathcal{B}_r^X(x^\dagger)$ with radius $r > 0$ and center $x^\dagger$ such that for each sequence $\{x_n\}_{n=1}^\infty \subset \mathcal{B}_r^X(x^\dagger) \cap \mathcal{D}(F)$ the implication

$$\lim_{n\to\infty} \|F(x_n) - F(x^\dagger)\|_Y = 0 \implies \lim_{n\to\infty} \|x_n - x^\dagger\|_X = 0$$

holds true. Otherwise $(*)$ is called **locally ill-posed** at $x^\dagger$.

Note that local well-posedness requires **local injectivity**.

We focus on nonlinear F and **local ill-posedness** at $x^\dagger$. Then

$$\|x - x^\dagger\|_X \le K\,\varphi(\|F(x) - F(x^\dagger)\|_Y) \quad \text{for all } \ x \in \mathcal{B}_r^X(x^\dagger) \cap \mathcal{D}(F)$$

cannot hold for any constants $K, r > 0$ and **index functions** φ.

However, such **conditional stability estimates** can hold if $\|x - x^\dagger\|_X$ is substituted by weaker norms $\|x - x^\dagger\|_{-a}$ $(a > 0)$ in the context of **Hilbert scales** $\{X_\tau\}_{\tau \in \mathbb{R}}$ generated by a densely defined, unbounded, linear, and self-adjoint operator $B\colon \mathcal{D}(B) \subset X \to X$ with $\|x\|_\tau := \|B^\tau x\|_X$ and $\mathcal{D}(B) = X_1$. $\|Bx\|_X \ge c_B \|x\|_X$ is valid for all $x \in X_1$ with constant $c_B > 0$.

A powerful tool in the **Hilbert scale** $\{X_\tau\}_{\tau\in\mathbb{R}}$ generated by B is the **interpolation inequality**, which attains for $-a < t \le p$ the form

$$\|x\|_t \le \|x\|_{-a}^{\frac{p-t}{p+a}} \|x\|_p^{\frac{t+a}{p+a}} \qquad \text{for all} \quad x \in X_p.$$

Assumption 1

- The operator F is weak-to-weak sequentially continuous.
- The domain $\mathcal{D}(F)$ is a convex and closed subset of X.
- $\mathcal{D} = \mathcal{D}(F) \cap \mathcal{D}(B) \neq \emptyset$.
- $x^\dagger \in \mathcal{D}(F)$ is the uniquely determined solution to $(*)$.
- Regularized solutions x_α^δ are minimizers of

$$T_\alpha^\delta(x) := \|F(x) - y^\delta\|_Y^2 + \alpha\|Bx\|_X^2 \to \min, \quad \text{s.t. } x \in \mathcal{D}(F),$$

consequently $x_\alpha^\delta \in \mathcal{D} = \mathcal{D}(F) \cap X_1$.

This assumption ensures the **existence** and **stability** of regularized solutions x_α^δ for all $\alpha > 0$.

Case distinction

(a) $x^\dagger \in X_p$ for some $p > 1$, which means that $\|Bx^\dagger\|_X < \infty$ and there is some source element $w \in X_\varepsilon$ $(\varepsilon > 0)$ such that $x^\dagger = B^{-1}w$. (undersmoothing penalty case)

(b) $x^\dagger \in X_1$, which means that $\|Bx^\dagger\|_X < \infty$, but $x^\dagger \notin X_{1+\varepsilon}$ for all $\varepsilon > 0$. (borderline case)

(c) $x^\dagger \in X_p$ for some $0 < p < 1$, but $x^\dagger \notin X_1$, which means that $\|Bx^\dagger\|_X = \infty$. (oversmoothing penalty case).

Outline

Convergence

By definition of the Tikhonov functional we have $x_\alpha^\delta \in X_1$, but only in the cases (a) and (b) one can take profit of the inequality

$$T_\alpha^\delta(x_\alpha^\delta) \le T_\alpha^\delta(x^\dagger),$$

which implies for all $\alpha > 0$ that

$$\|x_\alpha^\delta\|_1 \le \sqrt{\|x^\dagger\|_1^2 + \frac{\delta^2}{\alpha}}.$$

In the case (c), however, due to $x^\dagger \notin X_1$ and hence $\|x^\dagger\|_1 = \infty$ we have no such uniform bounds of $\|x_\alpha^\delta\|_1$ from above. Evidently, in case (c), $\|x_\alpha^\delta\|_1 \to \infty$ as $\delta \to 0$ is necessary for convergence of the regularized solutions x_α^δ to $x^\dagger$.

Proposition 1 (convergence)

Let the regularization parameter $\alpha > 0$ fulfill the conditions

$$\alpha \to 0 \qquad \text{and} \qquad \frac{\delta^2}{\alpha} \to 0 \qquad \text{as} \qquad \delta \to 0.$$

Then we have under Assumption 1 and for cases (a) and (b) by setting $\alpha_n = \alpha(\delta_n)$ or $\alpha_n = \alpha(\delta_n, y^{\delta_n})$, $x_n = x_{\alpha_n}^{\delta_n}$, that for $\delta_n \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} \|x_n\|_1 = \|x^\dagger\|_1,$$

and

$$\lim_{n\to\infty} \|x_n - x^\dagger\|_\nu = 0 \qquad \text{for all} \qquad 0 \le \nu \le 1.$$

Corollary

Under the assumptions and for α-choices of Proposition 1 we have for cases (a) and (b) that the regularized solutions x_α^δ belong to the ball $\mathcal{B}_r^{X_\nu}(x^\dagger)$ for prescribed values $r > 0$ and $0 \le \nu \le 1$ whenever $\delta > 0$ is sufficiently small.

In general, in case (c) one cannot even show weak convergence of x_α^δ as $\delta \to 0$. Regularized solutions x_α^δ need not belong to a ball $\mathcal{B}_r^X(x^\dagger)$ with small radius $r > 0$ if $\delta > 0$ is sufficiently small. Under stronger conditions, however, convergence can be the consequence of proven convergence rates.

Outline

Convergence rates under conditional stability estimates

Assumption 2

Let $a > 0,\ 0 < \gamma \le 1$, and let the conditional stability estimates

$$\|x - x^\dagger\|_{-a} \le K(\varrho)\,\|F(x) - F(x^\dagger)\|_Y^\gamma \quad \text{for all } \ x \in \mathcal{B}_\varrho^{X_1}(0) \cap \mathcal{D}(F)$$

hold, where constants $K(\varrho) > 0$ are supposed to exist for all radii $\varrho > 0$.

Extension to general concave index function φ as

$$\|x - x^\dagger\|_{-a} \le K(\varrho)\,\varphi(\|F(x) - F(x^\dagger)\|_Y) \quad \text{for all } \ x \in \mathcal{B}_\varrho^{X_1}(0) \cap \mathcal{D}(F)$$

was recently outlined in $\triangleright$ WERNER/H. 2019.

Proposition 2 (undersmoothing penalties) $\triangleright$ EGGER/H. IP 2018

Under the Assumptions 1 and 2 and for $x^\dagger \in X_p$ with $1 < p \le a+2$ we have the rate of convergence of regularized solutions $x_\alpha^\delta \in \mathcal{D}(F) \cap \mathcal{D}(B)$ to the solution $x^\dagger \in \mathcal{D}(F) \cap X_p$ as

$$\|x_\alpha^\delta - x^\dagger\|_X = \mathcal{O}\left(\delta^{\frac{\gamma p}{p+a}}\right) \qquad \text{as} \quad \delta \to 0,$$

provided that the regularization parameter $\alpha = \alpha(\delta)$ is chosen a priori as

$$\alpha(\delta) \sim \delta^{2-2\gamma\frac{p-1}{p+a}}.$$

For that choice of the regularization parameter we have

$$\alpha(\delta) \to 0 \qquad \text{and} \qquad \frac{\delta^2}{\alpha(\delta)} \to 0 \qquad \text{as} \qquad \delta \to 0.$$

Proposition 3 ▷ CHENG/YAMAMOTO IP 2000

Under the Assumptions 1 and 2 and for $x^\dagger \in X_1$ we have the rate of convergence of regularized solutions $x_\alpha^\delta \in \mathcal{D}(F) \cap \mathcal{D}(B)$ to the solution $x^\dagger \in \mathcal{D}(F) \cap \mathcal{D}(B)$ as

$$\|x_\alpha^\delta - x^\dagger\|_X = \mathcal{O}\left(\delta^{\frac{\gamma}{1+a}}\right) \qquad \text{as} \quad \delta \to 0,$$

if the regularization parameter $\alpha = \alpha(\delta)$ is chosen a priori as

$$\alpha(\delta) \sim \delta^2.$$

This result is also valid for borderline case.

For that choice of the regularization parameter we have for constants $0 < \underline{c} \le \overline{c} < \infty$

$$\alpha(\delta) \to 0 \qquad \text{and} \qquad \underline{c} \le \frac{\delta^2}{\alpha(\delta)} \le \overline{c} \qquad \text{as} \qquad \delta \to 0.$$

Assumption 3

Let $a, r > 0,\ 0 < \gamma \leq 1$, and let the conditional stability estimate

$$\|x - x^\dagger\|_{-a} \leq K(r)\,\|F(x) - F(x^\dagger)\|_Y^\gamma \quad \text{for all} \quad x \in \mathcal{B}_r^X(x^\dagger) \cap \mathcal{D}(F)$$

hold, where the constant $K(r) > 0$ depends on the prescribed r.

As a consequence of the above Corollary Proposition 2 remains valid if Assumption 2 is substituted by Assumption 3.

Outline

Convergence rates in case of oversmoothing penalties

Assumption 4

- Let $a > 0, r > 0$ and let $x^\dagger \in \mathrm{int}(\mathcal{D}(F))$ with $\mathcal{B}_r^X(x^\dagger) \subset \mathcal{D}(F)$.
- Let there exist constants $0 < \underline{K} \le \overline{K} < \infty$ such that

$$\underline{K}\,\|x-x^\dagger\|_{-a} \le \|F(x)-F(x^\dagger)\|_Y \quad \text{for all} \quad x \in \mathcal{D}(F)\cap X_1 \quad (\$L)$$

and

$$\|F(x)-F(x^\dagger)\|_Y \le \overline{K}\,\|x-x^\dagger\|_{-a} \ \text{ for all } \ x \in \mathcal{B}_r^X(x^\dagger)\cap X_1. \ (\$R)$$

Theorem (case of oversmoothing penalties)

Let $x^\dagger \in X_p$ for some $0 < p < 1$, but assume $x^\dagger \notin X_1$. Under Assumptions 1 and 4 we then have the rate of convergence of regularized solutions to the exact solution as

$$\|x_{\alpha_*}^\delta - x^\dagger\|_X = \mathcal{O}\left(\delta^{\frac{p}{p+a}}\right) \qquad \text{as} \quad \delta \to 0,$$

if the regularization parameter is chosen a priori as

$$\alpha_* = \alpha(\delta) = \delta^{2-2\frac{p-1}{p+a}}.$$

For that choice of the regularization parameter we have

$$\alpha(\delta) \to 0 \qquad \text{and} \qquad \frac{\delta^2}{\alpha(\delta)} \to \infty \qquad \text{as} \qquad \delta \to 0.$$

Sketch of a proof: For simplicity we set $E := \|x^\dagger\|_p$.
To prove the rate result it is sufficient to show that, for sufficiently small $\delta > 0$, there are two constants $K > 0$ and $\tilde{E} > 0$ such that the inequalities

$$\|x_{\alpha_*}^\delta - x^\dagger\|_{-a} \le K\delta \qquad (I1)$$

and

$$\|x_{\alpha_*}^\delta - x^\dagger\|_p \le \tilde{E} \qquad (I2)$$

hold. Then the rate follows directly from

$$\|x_{\alpha_*}^\delta - x^\dagger\|_X \le \|x_{\alpha_*}^\delta - x^\dagger\|_{-a}^{\frac{p}{a+p}} \|x_{\alpha_*}^\delta - x^\dagger\|_p^{\frac{a}{a+p}} \le K^{\frac{p}{a+p}} \tilde{E}^{\frac{a}{a+p}} \delta^{\frac{p}{a+p}},$$

which is valid, for sufficiently small $\delta > 0$, as a consequence of $(I1)$, $(I2)$ and of the interpolation inequality for the Hilbert scale.

Now it remains to prove $(I1)$ and $(I2)$.

As an essential tool for the proof we use **auxiliary elements** x_α, which are, for all $\alpha > 0$, the uniquely determined minimizers over all $x \in X$ of the **artificial Tikhonov functional**

$$T_{-a,\alpha}(x) := \|x - x^\dagger\|_{-a}^2 + \alpha \|Bx\|_X^2.$$

The mapping $x^\dagger \mapsto x_\alpha$ is a variant of **proximal operator.**

Note that the elements x_α are independent of the noise level δ and belong by definition to X_1, in strong contrast to $x^\dagger \notin X_1$.

Lemma ▷ H./MATHÉ IP 2018

Let $\|x^\dagger\|_p = E$ and x_α be the minimizer of the functional $T_{-a,\alpha}$. Given

$$\alpha_* = \alpha(\delta) = \delta^{2-2\frac{p-1}{p+a}} > 0,$$

the resulting element x_{α_*} obeys the bounds

$$\|x_{\alpha_*} - x^\dagger\|_X \le E\delta^{p/(a+p)}, \qquad (I3)$$

$$\|B^{-a}(x_{\alpha_*} - x^\dagger)\|_X \le E\delta, \qquad (I4)$$

$$\|Bx_{\alpha_*}\|_X \le E\delta^{(p-1)/(a+p)} = E\frac{\delta}{\sqrt{\alpha_*}} \qquad (I5)$$

and

$$\|x_{\alpha_*} - x^\dagger\|_p \le E. \qquad (I6)$$

Due to $(I3)$ we have $\|x_{\alpha_*} - x^\dagger\|_X \to 0$ as $\delta \to 0$.
Hence by Assumption 4 ($x^\dagger$ is an interior point of $\mathcal{D}(F)$)
we have that, for sufficiently small $\delta > 0$ the element x_{α_*}
belongs to $\mathcal{B}_r^X(x^\dagger) \subset \mathcal{D}(F)$ and moreover with $x_{\alpha_*} \in X_1$
the right-hand side inequality ($\$R$) applies for $x = x_{\alpha_*}$.

Instead of the usual regularizing property $T_\alpha^\delta(x_\alpha^\delta) \le T_\alpha^\delta(x^\dagger)$,
which is missing in case of oversmoothing penalties, we use

$$T_{\alpha_*}^\delta(x_{\alpha_*}^\delta) \le T_{\alpha_*}^\delta(x_{\alpha_*}) \qquad (MP)$$

as a helpful minimizing property for the Tikhonov functional.

Using the minimizing property (MP) it is enough to bound $T^{\delta}_{\alpha_*}(x_{\alpha_*})$ by $\overline{C}^2\delta^2$ with $\overline{C} := \left((\overline{K}E+1)^2+E^2\right)^{1/2}$ in order to obtain the estimates

$$\|F(x^{\delta}_{\alpha_*}) - y^{\delta}\|_Y \le \overline{C}\delta$$

and

$$\|Bx^{\delta}_{\alpha_*}\|_X \le \overline{C}\frac{\delta}{\sqrt{\alpha_*}}.$$

Since the inequality ($R) applies for $x = x_{\alpha_*}$ and sufficiently small $\delta > 0$, we can estimate with (I5) for such δ as follows:

$$
\begin{aligned}
T^{\delta}_{\alpha_*}(x_{\alpha_*}) &\le \left(\|F(x_{\alpha_*}) - F(x^\dagger)\|_Y + \|F(x^\dagger) - y^\delta\|_Y\right)^2 + \alpha_* \|Bx_{\alpha_*}\|_X^2 \\
&\le \left(\overline{K}\|x_{\alpha_*} - x^\dagger\|_{-a} + \delta\right)^2 + E^2 \alpha_* \delta^{2(p-1)/(a+p)} \\
&\le \left(\overline{K}E\delta + \delta\right)^2 + E^2\delta^2 \\
&= \left((\overline{K}E + 1)^2 + E^2\right)\delta^2.
\end{aligned}
$$

Based on this we can show that $(I1)$ is valid for some $K > 0$. Here, we use the left-hand inequality ($L) of Assumption 4, which applies for $x = x_{\alpha_*}^\delta \in \mathcal{D}(F) \cap X_1$, and we find

$$\begin{aligned} \|x_{\alpha_*}^\delta - x^\dagger\|_{-a} &\le \frac{1}{\underline{K}} \|F(x_{\alpha_*}^\delta) - F(x^\dagger)\|_Y \\ &\le \frac{1}{\underline{K}} \left(\|F(x_{\alpha_*}^\delta) - y^\delta\|_Y + \|F(x^\dagger) - y^\delta\|_Y \right) \\ &\le \frac{1}{\underline{K}} \left(\overline{C}\delta + \delta \right) = \frac{1}{\underline{K}} \left(\overline{C} + 1 \right) \delta = K\delta. \end{aligned}$$

Hence, we derive $K := \frac{1}{\underline{K}} \left(\overline{C} + 1 \right)$ for the constant in $(I1)$.

Finally, we still have to show the existence of a constant $\tilde{E} > 0$ such that the inequality $(I2)$ holds.

By exploiting the triangle inequality we find that

$$\|B(x_{\alpha_*}^\delta - x_{\alpha_*})\|_X \le \|Bx_{\alpha_*}^\delta\|_X + \|Bx_{\alpha_*}\|_X \le (\overline{C} + E)\frac{\delta}{\sqrt{\alpha_*}}.$$

Using the interpolation inequality we can estimate further as

$$\|x_{\alpha_*}^\delta - x_{\alpha_*}\|_p \le \|x_{\alpha_*}^\delta - x_{\alpha_*}\|_1^{\frac{a+p}{a+1}} \|x_{\alpha_*}^\delta - x_{\alpha_*}\|_{-a}^{\frac{1-p}{a+1}}$$

$$\le \left((\overline{C}+E)\frac{\delta}{\sqrt{\alpha_*}}\right)^{\frac{a+p}{a+1}} \left(\|x_{\alpha_*}^\delta - x^\dagger\|_{-a} + \|x^\dagger - x_{\alpha_*}\|_{-a}\right)^{\frac{1-p}{a+1}}$$

$$\le \left((\overline{C}+E)\frac{\delta}{\sqrt{\alpha_*}}\right)^{\frac{a+p}{a+1}} ((K+E)\delta)^{\frac{1-p}{a+1}}$$

$$\left((\overline{C}+E)\delta^{(p-1)/(a+p)}\right)^{\frac{a+p}{a+1}} ((K+E)\delta)^{\frac{1-p}{a+1}} =: \bar{E}.$$

Consequently, we have now

$$\|x_{\alpha_*}^\delta - x^\dagger\|_p \le \|x_{\alpha_*}^\delta - x_{\alpha_*}\|_p + \|x_{\alpha_*} - x^\dagger\|_p \le \bar{E} + E =: \tilde{E}.$$

This shows ($I2$) and thus completes the proof. □

Outline

Example (exponential growth model)

We aim at identifying the time dependent growth rate $x(t)$ $(0 \le t \le T)$ from observations of the size $y(t)$ $(0 \le t \le T)$ of a population with $y(0) = y_0 > 0$ such that the problem

$$y'(t) = x(t)\,y(t) \quad (0 \le t \le T), \qquad y(0) = y_0\,,$$

is satisfied.

For $X = Y = L^2(0,T)$ the forward operator attains the form

$$[F(x)](t) = y_0 \, \exp\left(\int_0^t x(\tau)d\tau\right) \quad (0 \le t \le T).$$

We note that the corresponding nonlinear operator equation $(*)$ is **locally ill-posed everywhere** in X.

Moreover, the operator F is continuously Fréchet differentiable on the whole Hilbert space $L^2(0,1)$ and has the derivative

$$[F'(x^\dagger)h](t) = [F(x^\dagger)](t) \int_0^t h(\tau)d\tau \quad (0 \le t \le T), \quad h \in L^2(0,T).$$

One easily verifies that

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\|_Y \le \hat{K}\, \|F(x) - F(x^\dagger)\|_Y \|x - x^\dagger\|_X$$

holds with some constant $\hat{K} > 0$ for all $x \in X$. For $\eta := r\hat{K} < 1$

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\|_Y \le \eta\, \|F(x) - F(x^\dagger)\|_Y$$

is satisfied with $0 < \eta < 1$ and yields for such r and $x \in \mathcal{B}_r^X(x^\dagger)$

$$K_{low}\, \|F'(x^\dagger)(x - x^\dagger)\|_Y \le \|F(x) - F(x^\dagger)\|_Y \le K_{up}\, \|F'(x^\dagger)(x - x^\dagger)\|_Y$$

with $K_{low} = 1/(1+\eta)$ and $K_{up} = 1/(1-\eta)$.

Now set for simplicity $T := 1$. In order to generate a Hilbert scale $\{X_\tau\}_{\tau\in\mathbb{R}}$, we exploit the simple integration operator

$$[Jh](t) := \int_0^t h(\tau)d\tau \quad (0 \le t \le 1)$$

of Volterra-type mapping in $X = Y = L^2(0,1)$ and set

$$B := (J^*J)^{-1/2}.$$

By considering the Riemann-Liouville fractional integral operator J^p and its adjoint $(J^*)^p = (J^p)^*$ for $0 < p \le 1$ we have with $X_p = \mathcal{D}(B^p) = \mathcal{R}((J^*J)^{p/2}) = \mathcal{R}((J^*)^p)$

$$X_p = \begin{cases} H^p(0,1) & \text{for} \quad 0 < p < \frac{1}{2} \\ \{x \in H^{\frac{1}{2}}(0,1) : \int\limits_0^1 \frac{|x(t)|^2}{1-t} dt < \infty\} & \text{for} \quad p = \frac{1}{2} \\ \{x \in H^p(0,1) :\ x(1) = 0\} & \text{for} \quad \frac{1}{2} < p \le 1 \end{cases},$$

where the fractional Sobolev spaces $H^p(0,1)$ occur.

Now we have that

$$\|Jh\|_Y = \|(J^*J)^{1/2}h\|_X = \|B^{-1}h\|_X = \|h\|_{-1} \quad \text{for all} \quad h \in X$$

and that there are constants $0 < \underline{c} \le \overline{c} < \infty$ such that

$$\underline{c} \le [F(x^\dagger)](t) \le \overline{c} \qquad (0 \le t \le 1)$$

for the multiplier function in $F'(x^\dagger)$. Thus we have for all $x \in X$

$$\underline{c}\,\|x - x^\dagger\|_{-1} \le \|F'(x^\dagger)(x - x^\dagger)\|_Y \le \overline{c}\,\|x - x^\dagger\|_{-1}$$

and consequently estimates ($\$L$) as well as ($\$R$) with $a = 1$ and $\underline{K} = \underline{c}K_{low}, \overline{K} = \overline{c}K_{up}$, but both restricted to $x \in \mathcal{B}_r^X(x^\dagger)$ and sufficiently small $r > 0$.

Example (autoconvolution)

With the same Hilbert scale generator B based on J we can consider the **autoconvolution operator** in $X = Y = L^2(0,1)$

$$[F(x)](s) = \int_0^t x(s-t)x(t)dt \quad (0 \le s \le 1),$$

where $\mathcal{D}(F) = X$ and we have the Fréchet derivative

$$[F'(x)h](s) = 2\int_0^s x(s-t)\,h(t)dt \quad (0 \le s \le 1, \;\; h \in X)$$

satisfying for all $x \in X$ the estimate

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\|_Y = \|F(x - x^\dagger)\|_Y \le \|x - x^\dagger\|_X^2.$$

For the specific solution $x^\dagger(t) = 1 \;\; (0 \le t \le 1)$ we have that

$$\|F'(x^\dagger)h\|_Y = 2\|Jh\|_Y = 2\|B^{-1}h\|_X = \|h\|_{-1} \quad \text{for all} \quad h \in X.$$

Using the interpolation inequality $\|h\|_X^2 \le \|h\|_{-1}\|h\|_1$ we derive

$$\|x-x^\dagger\|_{-1} \le \frac{1}{2}\|F(x)-F(x^\dagger)-F'(x^\dagger)(x-x^\dagger)\|_Y+\frac{1}{2}\|F(x)-F(x^\dagger)\|_Y$$

$$\le \frac{1}{2}\|F(x)-F(x^\dagger)\|_Y + \frac{1}{2}\|x-x^\dagger\|_1\|x-x^\dagger\|_{-1} \quad \text{for} \quad x-x^\dagger \in X_1$$

and for $\|x-x^\dagger\|_1 \le \kappa < 2$ even the conditional stability estimate

$$\|x-x^\dagger\|_{-1} \le \frac{1}{2-\kappa}\|F(x)-F(x^\dagger)\|_Y \quad \text{for all} \quad x-x^\dagger \in \mathcal{B}_\kappa^{X_1}(0).$$

Note that $x^\dagger \in X_p$ if an only if $p < 0.5 \;\Rightarrow\; x_\alpha^\delta - x^\dagger \notin \mathcal{B}_\kappa^{X_1}(0)$.

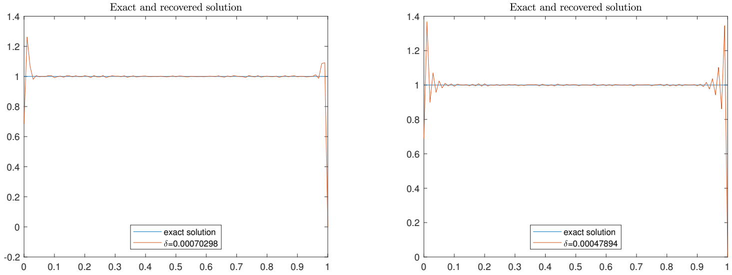

Figure: $x^\dagger \equiv 1$ and regularized solutions x_α^δ for varying noise levels δ

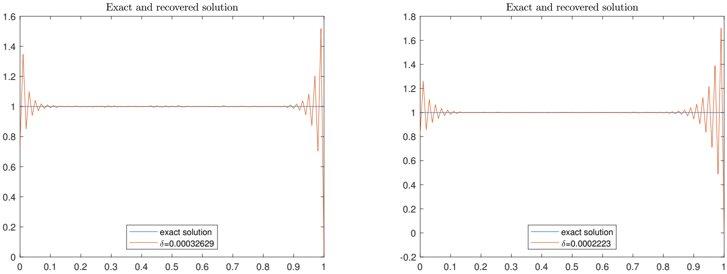

Figure: $x^\dagger \equiv 1$ and regularized solutions x_α^δ for varying noise levels δ