On the Qualitative Approach in Inverse Scattering for the Time Dependent Wave Equation

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RUTGERS

Inverse Scattering

Popular approaches to the inverse scattering problem for acoustic/electromagnetic/elatic waves in the frequency domain:

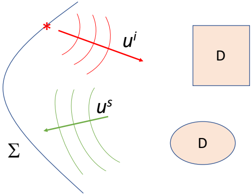
1. Linearization: Ignores multiple scattering and hence model may be incorrect.
2. Nonlinear Optimization: Typically reconstruct all the unknowns. Possibly little date, but good a priori information. Convergence of Newton's Method for inverse scattering problem is not fully established.
3. Data Driven Models: Being developed.
4. Qualitative Method: No a priori information, but needs a lot of data. Only determines support of scattering object. It is mathematically rigorous with correct model.

A. Kirsch and N. Grinberg (2008), *The Factorization Method for Inverse Problems, Oxford University*.

F. Cakoni and D. Colton and H. Haddar (2016), *Inverse Scattering Theory and Transmission Eigenvalues, CBMS-NSF, SIAM Publication*.

Qualitative Methods

Consider a family of interrogating incident fields $u^i(x; y)$ for an array of transmitters $y \in \Sigma$. Measure the corresponding scattered field $u^s(x; y)$ at an array of receivers $x \in \Sigma$.



The relative scattering operator

$$(Ng)(x) := \int_\Sigma g(y) u^s(x; y) ds_y \qquad x \in \Sigma$$

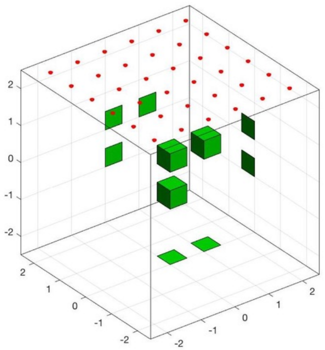
Determine scatters' support D from a knowledge of N.

In fact one test if $\varphi_z \in \mathrm{Range}(N)$ for particular functions φ_z depending on z sampling a region containing D

Qualitative Methods

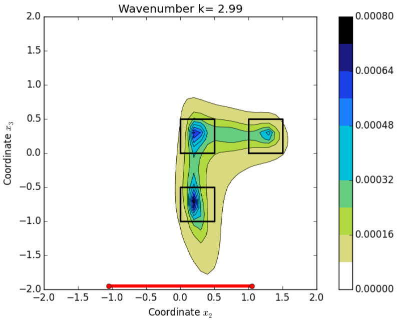
In principle one could use a single frequency. In this case for limited aperture data and sparse array the method may provide a poor reconstruction of D.

For example, consider the following scatterers and measurement array where the point sources and receivers are at the point in the grid above the scatterers.



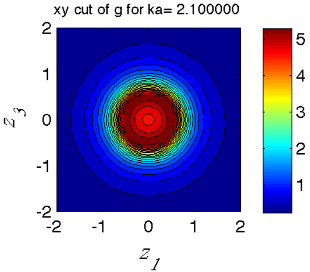
The Linear Sampling Method

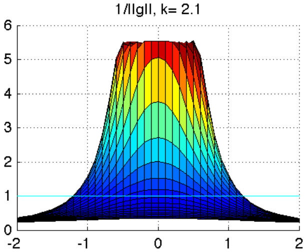
Below are the cross sections in the plane $x_1 = 0.25$ corresponding to $k = 2.99$ and $k = 6.03$. Note that the cross section for $k = 6.03$ misses the lower scatterer.

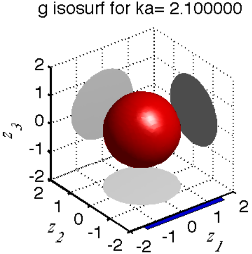


Examples of Reconstruction

$D := B_1$, $n = 16$, k is not TE

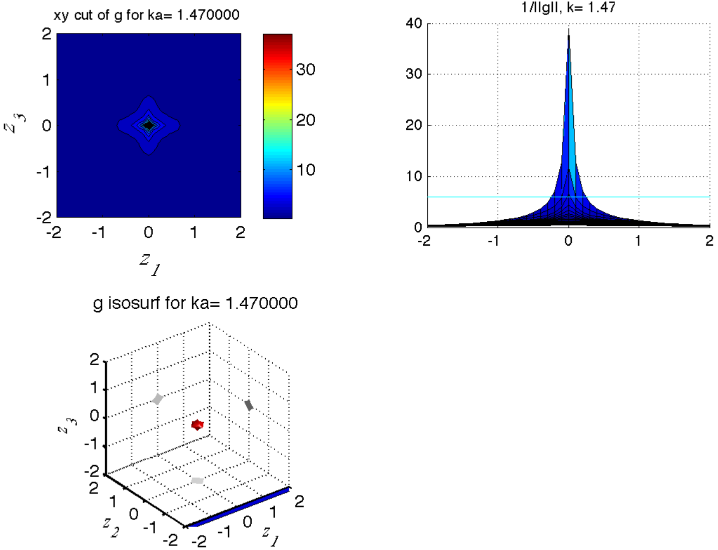






Examples of Reconstruction

$D := B_1$, $n = 16$, k a TE



Qualitative Methods

Problems with Qualitative Methods at a fixed frequency

1. The "good" frequency is not known a priori.
2. Dense spacial measurements of a large aperture are needed for achieving reasonably good reconstructions.

How can these issues be remedied?

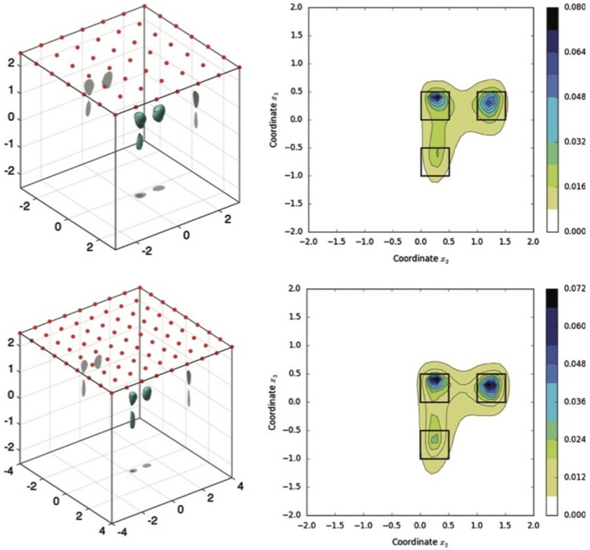
- Approach 1: Use multifrequency data.

  B. Guzina, F. Cakoni, C. Bellis (2010), R. Griesmaier, C. Schmiedecke (2017),

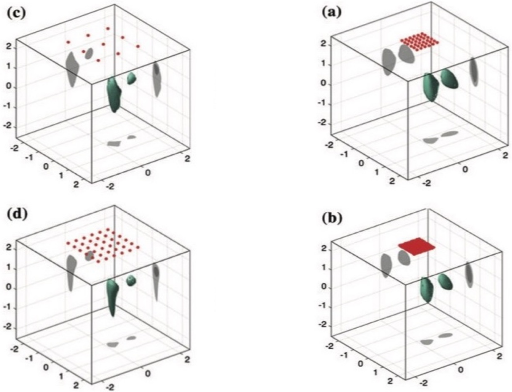
- Approach 2: Use directly time domain data.

  D. R. Luke, R. Potthast (2006), Q. Chen, H. Haddar, A. Lechleiter, P. Monk (2010), H. Haddar, A. Lechleiter, S. Marmorat (2014), Y. Guo, P. Monk, D. Colton (2013), (2015), F. Cakoni, J. Rezac (2017), L. Oksanen (2013), L. Bourgeois, D. Ponomarev, J. Darte (2019), M. Ikehata, (a series of papers on the enclosure method)

Examples of reconstruction with time domain data



Examples of reconstruction with time domain data



Examples taken from

Y. Guo, P. Monk, D. Colton (2016), The linear sampling method for sparse small aperture data, *Applicable Analysis*.

The Outline

Solvability of the time domain interior transmission problem

The beginning of the end story: justification of the linear sampling method for inhomogeneous media

F. Cakoni, P. Monk, V. Selgas (to appear), Analysis of the linear sampling method for imaging penetrable obstacles in the time domain, *Analysis & PDEs*.

Toward a time domain factorization method

The beginning of an open story: the derivation of a mathematically justified time domain qualitative approach.

F. Cakoni, H. Haddar, A. Lechleiter (2019), On the factorization method for a far field inverse scattering problem in the time domain, *SIAM J. Math. Anal.*

Scattering in the Time Domain

Let X be a Hilbert space and $f(t)$ is such that $e^{-\sigma t}f(t) \in L^1(\mathbb{R}, X)$ for some $\sigma > 0$. Define the Fourier-Laplace transform by

$$\mathcal{L}[f](s) := \int_{-\infty}^{\infty} e^{ist} f(t) dt, \qquad s \in \mathbb{C}_\sigma$$

where $\mathbb{C}_\sigma := \{s \in \mathbb{C},\ \Im(s) > \sigma\}$. For $m \in \mathbb{R}$ define the Hilbert space

$$H^m_\sigma(\mathbb{R}, X) := \left\{ f : \int_{-\infty+i\sigma}^{\infty+i\sigma} |s|^{2m} \left\| \mathcal{L}[f](s) \right\|_X ds < \infty \right\}$$

endowed with the norm

$$\|f\|_{H^m_\sigma(\mathbb{R},X)} = \left(\int_{-\infty+i\sigma}^{\infty+i\sigma} |s|^{2m} \left\| \mathcal{L}[f](s) \right\|_X ds \right)^{1/2} .$$

Scattering in the Time Domain

Supp $(1-n) = \overline{D}$ $\quad \partial D$ Lipshitz $\quad f := (1-n)\partial_{tt}^2 u^i$

The scattered field u^s satisfy:

$$n(x)\partial_{tt}^2 u^s - \Delta u^s = f \quad \text{in } \mathbb{R}^3 \quad \text{for } t > 0$$

$$u^s = 0 \quad \text{in } \mathbb{R}^3 \quad \text{for } t \le 0.$$

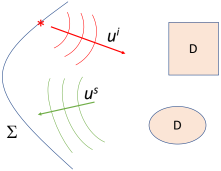
$n(x) = c_0^2 / c^2(x) \ge n_0 > 0$ is piecewise smooth.

The linear mapping $\mathcal{G} : f \mapsto u^s$ is bounded as

$$H_\sigma^{m+1}(\mathbb{R}, H^{-1}(\mathbb{R}^3)) \to H_\sigma^m(\mathbb{R}, H^1(\mathbb{R}^3))$$

$$H_\sigma^m(\mathbb{R}_+, L^2(\mathbb{R}^3)) \to H_\sigma^m(\mathbb{R}_+, H^1(\mathbb{R}^3)).$$

Scattering in the Time Domain



Let χ denote a smooth function of compact support on $(0, \infty)$. Then the incident field u^i is defined by

$$u^i(t, x; y) = \frac{\chi(t - |x - y|)}{4\pi|x - y|}.$$

Inverse Problem

Let Σ be a portion of an analytic surface lying outside D. Assume we know the scattered field $u^s(t, x; y)$ for $t > 0$, $x \in \Sigma$, corresponding to the incident field $u^i(t, x; y)$, for $y \in \Sigma$. Determined D.

The Near Field Operator

Define the near field operator $\mathcal{N} : H^m_\sigma(\mathbb{R}_+, L^2(\Sigma)) \to H^m_\sigma(\mathbb{R}_+, L^2(\Sigma))$

$$(\mathcal{N}\varphi)(t,x) = \int_\Sigma \int_{-\infty}^{t} u^s(\tau, x; y)\varphi(t-\tau, y)d\tau ds_y$$

Then

$$\mathcal{N}\varphi = \gamma_\Sigma U_\varphi \qquad \text{where} \qquad U_\varphi := \mathcal{G}\left[(1-n)\partial^2_{tt}(\mathcal{S}\varphi)\right]$$

where retarded single layer potential $\mathcal{S}$ defined by

$$(\mathcal{S}\varphi)(t,x) = \int_\Sigma \int_{-\infty}^{t} u^i(\tau, x; y)\varphi(t-\tau, y)d\tau ds_y.$$

- $\mathcal{N}$ is injective with dense range
- $\mathcal{S}\left[H^m_\sigma(\mathbb{R}_+, L^2(\Sigma))\right]$ is dense in $H^{m+2}_\sigma(\mathbb{R}_+, H^1(\mathbb{R}^3))$

The Near Field Equation

$$n(x)\partial_{tt}^2 U_\varphi - \Delta U_\varphi = (1-n)\partial_{tt}^2(\mathcal{S}\varphi) \quad \text{in } \mathbb{R}^3 \quad \text{for } t>0$$

and we also have $$\partial_{tt}^2(\mathcal{S}\varphi) - \Delta(\mathcal{S}\varphi) = 0$$

We consider the near field equation

$$(\mathcal{N}\varphi_z)(t,x) = \Phi_z(x,t), \qquad \text{for } z \in \mathbb{R}^3,\ x \in \Sigma$$

where

$$\Phi_z(x,t) := \frac{\xi(t-|x-z|)}{4\pi|x-z|}, \quad \tau \in \mathbb{R} \qquad \xi \in C^\infty(\mathbb{R}_+).$$

For $z \in D$, if φ_z solves the near field equation, we have that $\mathcal{N}\varphi_z$ and Φ_z will coincide up to the boundary of D for all $t > 0$.

The Near Field Equation

Thus $w := U_{\varphi_z} + \mathcal{S}\varphi_z$ and $v := \mathcal{S}\varphi_z$ satisfy the interior transmission problem in the time domain

$$
\begin{aligned}
\partial_{tt}^2 v - \Delta v &= 0 && \text{in} \quad \mathbb{R} \times D \\
n(x)\partial_{tt}^2 w - \Delta w &= 0 && \text{in} \quad \mathbb{R} \times D \\
w - v &= \Phi_z && \text{on} \quad \mathbb{R} \times \partial D \\
\partial_\nu w - \partial_\nu v &= \partial_\nu \Phi_z && \text{on} \quad \mathbb{R} \times \partial D \\
w = v &= 0 && \text{in } D \quad \text{for} \quad t \leq 0.
\end{aligned}
$$

Solvability of the interior transmission problem in the time domain was an open problem until now

Interior Transmission Problem

Time-Domain ITP

$$\begin{aligned} n\frac{\partial^2 w}{\partial t^2} - \Delta w &= F && \text{in } \mathbb{R}\times D \\ \frac{\partial^2 v}{\partial t^2} - \Delta v &= 0 && \text{in } \mathbb{R}\times D \\ w &= v && \text{on } \mathbb{R}\times \partial D \\ \frac{\partial w}{\partial \nu} &= \frac{\partial v}{\partial \nu} && \text{on } \mathbb{R}\times \partial D \\ w = v &= 0 && \text{in } D,\ t\le 0 \end{aligned}$$

Fourier-Domain ITP

$$\begin{aligned} \Delta \hat{w} + s^2 n \hat{w} &= \hat{F} && \text{in } D \\ \Delta \hat{v} + s^2 \hat{v} &= 0 && \text{in } D \\ \hat{w} &= \hat{v} && \text{on } \partial D \\ \frac{\partial \hat{w}}{\partial \nu} &= \frac{\partial \hat{v}}{\partial \nu} && \text{on } \partial D \end{aligned}$$

Laplace-Fourier Transform

$$\hat{f} := \mathcal{L}[f](s) = \int_{\mathbb{R}} e^{ist} f(t)\, dt, \qquad s \in \mathbb{C}_\sigma := \{s \in \mathbb{C} : \Im(s) > \sigma, \sigma > 0\}$$

Time Domain Versus Frequency Domain

The relationship between resolvent estimates in the frequency domain and solvability of the interior transmission problem in the time-domain is arrived through the following lemma.

Lemma (Lubich)

Assume the mapping $s \in \mathbb{C}_\sigma \mapsto \hat{A}_s \in \mathcal{B}(X, Y)$, $\sigma > 0$ is analytic

$$\text{and} \qquad \|\hat{A}_s\|_{\mathcal{B}(X,Y)} \leq C\,|s|^r \quad \text{for a.e. } s \in \mathbb{C}_\sigma \text{ and some } r \in \mathbb{R}$$

Set

$$a(t) = \frac{1}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma} e^{-ist}\hat{A}_s\,ds, \qquad \text{and} \qquad Ag = \int_{-\infty}^{\infty} a(t)\,g(\cdot - t)\,dt.$$

Then, $A : H_\sigma^{m+r}(\mathbb{R}, X)$ to $H_\sigma^m(\mathbb{R}, Y)$ is bounded for all $m \in \mathbb{R}$.

Transmission Eigenvalues

The mail difficulty in establishing the solvability of the ITP was to determine the location in the complex plane of transmission eigenvalues in the frequency domain.

The transmission eigenvalue problem in the frequency domain is to find s such that there exists a nontrivial solution $\hat{w}, \hat{v} \in L^2(D)$, $\hat{w} - \hat{v} \in H_0^2(D)$ such that

$$\begin{aligned} \Delta \hat{v} + s^2 \hat{v} &= 0 && \text{in} \quad D \\ \Delta \hat{w} + s^2 n(x) \hat{w} &= 0 && \text{in} \quad D \\ \hat{w} &= \hat{v} && \text{on} \quad \partial D \\ \partial_\nu \hat{w} &= \partial_\nu \hat{v} && \text{on} \quad \partial D \end{aligned}$$

F. CAKONI, D. COLTON AND H. HADDAR (2016), Transmission Eigenvalues and Inverse Scattering Theory, CBMS-NSF, SIAM Publications, **88**.

Solvability of ITP in Frequency Domain

ITP can be viewed as inverting the operator $M(s)$

$$M(s) := \mathcal{N}_{s,n} - \mathcal{N}_{s,1}$$

where $\mathcal{N}_{s,q}$ is the Dirichlet-to-Neuman operator for

$$\Delta u + s^2 q u = 0 \qquad \text{in } D$$

Assumptions: ∂D and n are piece-wise smooth and $n - 1$ have a fixed sign in a neighborhood of ∂D.

$M(s) : H^{-1/2+\tau}(\partial D) \to H^{1/2+\tau}(\partial D)$, $0 \leq \tau \leq 1$ is Fredholm operator with index zero and analytic in $\mathbb{C}$ (except for a discrete real set).

F. CAKONI, D. GINTIDES, H. HADDAR (2010), SYLVESTER (2012), H-M NGUYEN (2017), F. CAKONI, R. KRESS (2017)

Location of Transmission Eigenvalues

This is a problem with a perplexing structure.

Example

Let $n(x) = 4/9$ and $D := \{x : |x| < 1\}$. Then s is a TE $\Longleftrightarrow$

$$\sin^3\left(\frac{s}{3}\right)\left[3 + 2\cos\frac{2s}{3}\right] = 0$$

infinitely many complex transmission eigenvalues exist

Assume that $D := \{x : |x| < 1\}$, and $n \in C^2(\overline{D})$ and

$$n(1) = 1,\ \int_0^1 \sqrt{n(\theta)}d\theta \neq 1 \text{ and } n''(1) \neq 0$$

Then the transmission eigenvalues do not lie inside a fixed strip parallel to the real axis.

D. Colton, Y.J. Leung, S. Meng (2015)

Location of Transmission Eigenvalues

The icebreaker was a recent result by Vodev (2018).

Main Assumptions: $n \in C^\infty(\overline{D})$, ∂D is of C^∞-class, and $n \neq 1$ on ∂D

- Strip region for TE: there exists $\sigma_* > 0$ sufficiently large such that there exist no transmission eigenvalues in $\mathbb{C}_{\sigma_*} = \{s \in \mathbb{C}, \Im(s) > \sigma_*\}$.
- Frequency dependent resolvent estimates for high frequencies

Combining the high frequency estimates from Vodev (2018) with finite frequency estimates from Cakoni-Kress (2017), one can prove that that for $\Im(s) \geq \sigma^*$ for some σ^* large enough, $M(s)$ is invertible and the inverse satisfies

$$\|M(s)^{-1}\|_{H^{1/2+\tau}(\partial D)\to H^{-1/2+\tau}(\partial D)} \leq C|s|^{-1/2-\tau}, \qquad 0 \leq \tau \leq 1$$

G. Vodev (2018), High-frequency approximation of interior Dirichlet-to-Neumann map and applications to transmission eigenvalues, *Analysis and PDEs*.

The Interior Transmission Problem

This result thanks to the Lemma by Lubich, can be turned to a solvability theorem for the interior transmission problem in the time domain.

Theorem

Let $m \in \mathbb{R}$ and $\sigma > \sigma_*$. Given $h \in H^m_\sigma(\mathbb{R}, H^1(\partial D))$ and $g \in H^{m+5/2}_\sigma(\mathbb{R}, H^2(\partial D))$, the interior transmission problem in the time domain

$$\begin{aligned}
\partial^2_{tt} v - \Delta v &= 0 && \text{in} \quad \mathbb{R} \times D \\
n(x)\partial^2_{tt} w - \Delta w &= 0 && \text{in} \quad \mathbb{R} \times D \\
w - v &= g && \text{on} \quad \mathbb{R} \times \partial D \\
\partial_\nu w - \partial_\nu v &= h && \text{on} \quad \mathbb{R} \times \partial D \\
w = v &= 0 && \text{in } D \quad \text{for } t \leq 0.
\end{aligned}$$

has a unique solution $w, v \in H^m_\sigma(\mathbb{R}, L^2(D))$ which depends continuously on g and h.

The Linear Sampling Method in the Time Domain

Theorem

Let $\sigma > \sigma_*$ and $m \in \mathbb{R}$.

1. For $z \in D$ for every $\varepsilon > 0$, there exists some $\varphi_z^\varepsilon \in H_\sigma^m(\mathbb{R}, L^2(\Sigma))$ such that

$$\|\mathcal{N}\varphi_z^\varepsilon - \Phi_z\|_{H_\sigma^m(\mathbb{R}, H^{1/2}(\Sigma))} < \varepsilon$$

   and

$$\|\mathcal{S}\varphi_z^\varepsilon\|_{H_\sigma^{m+2}(\mathbb{R}, L^2(D))} < C \qquad \text{as } \varepsilon \to 0.$$

2. For $z \in \mathbb{R}^3 \setminus \overline{D}$, every sequence $\{\varphi_z^\varepsilon\}_{\varepsilon>0} \subset H_\sigma^m(\mathbb{R}, L^2(\Sigma))$ satisfying

$$\|\mathcal{N}\varphi_z^\varepsilon - \Phi_z\|_{H_\sigma^m(\mathbb{R}, H^{1/2}(\Sigma))} < \varepsilon$$

   is such that

$$\|\mathcal{S}\varphi_z^\varepsilon\|_{H_\sigma^{m+2}(\mathbb{R}, L^2(D))} \to \infty \qquad \text{as } \varepsilon \to 0.$$

Recall $\Phi_z(x,t) := \frac{\xi(t-|x-z|)}{4\pi|x-z|}, \quad \tau \in \mathbb{R} \qquad \xi \in C^\infty(\mathbb{R}_+)$.

Toward a rigorous sampling method

What do we have? $\mathcal{N} = \mathcal{G}\mathcal{S}$, Range($\mathcal{S}$) is dense in Dom($\mathcal{G}$),
$\Phi_z \in$ Range($\mathcal{G}$) $\Longleftrightarrow z \in D$

What do we want to have? $\mathcal{N}^{1/2}\mathcal{N}^{1/2} = \mathcal{S}^*\mathcal{T}\mathcal{S}$, $\mathcal{T}$ coercive.
Then Range($\mathcal{G}$) = Range($\mathcal{S}^*$) = Range($\mathcal{N}^{1/2}$).

Such approach is referred to as the Factorization Method.

A. Kirsch and N. Grinberg (2008), *The Factorization Method for Inverse Problems, Oxford University*.

Developing a time domain factorization method is a long standing open problem. Some initial efforts were made in

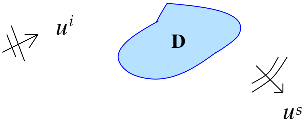
P. Tietäväinen (2011), A factorization method for the inverse scattering of the wave equation, *Ph.D. Thesis, Alto University*.

I will briefly discuss some recent progress toward a time domain factorization method.

F. Cakoni, H. Haddar and A. Lechleiter (2019), On the factorization method for a far field inverse scattering problem in the time domain , *SIAM Math Anal.*

The Scattering Problem

$D \subset \mathbb{R}^3$ is a Lipschitz domain such that $\mathbb{R}^3 \setminus \overline{D}$ is connected, and denote by $\mathbb{S}^2$ the unit sphere. The scattered field $u^s(x,t)$ satisfies:



$$
\begin{aligned}
\partial_{tt} u^s - \Delta u^s &= 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \times \mathbb{R} \\
u^s &= h \quad \text{on} \quad \partial D \times \mathbb{R} \\
u^s &= 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{D} \times (-\infty, 0)
\end{aligned}
$$

- $h := -u^i|_{\partial D}$ where the incident field $u^i(x,t)$ is a causal solution to the wave equation.
- $u_\infty(\xi, t) = \lim_{r \to \infty} r u^s(r\xi, r+t)$ for $\xi \in \mathbb{S}^2$ and $t \in \mathbb{R}$

$u_\infty : \mathbb{S}^2 \times \mathbb{R} \to \mathbb{R}$ is called the far field pattern of the causal scattered field u^s

see Friedlander 1962-1963-1964

Measured Data - Inverse Problem

Physical incident fields are traveling wavefront $u^i(x,t;\theta) := \delta(t - \theta \cdot x)$ with incident direction $\theta \in \mathbb{S}^2$. These are distributional causal solutions, $u^i = 0$ for $t < T$ with $-T > d := \sup_{x \in D} |x|$.

Inverse Problem

Reconstruct D from a knowledge of $u_\infty(\xi, t; \theta)$ on $\mathbb{S}^2 \times \mathbb{R} \times \mathbb{S}^2$

- The far field operator

$$(Fg)(\xi,t) := \int_{\mathbb{R}} \int_{\mathbb{S}^2} u_\infty(\xi, t-\tau; \theta) g(\theta,\tau)\, d\theta d\tau, \qquad g \in C_0^\infty(\mathbb{S}^2 \times \mathbb{R})$$

- Fg is the far field associated with the incident field

$$v_g(x,t) := \int_{\mathbb{R}} \int_{\mathbb{S}^2} \delta(t - \tau - \theta \cdot x) g(\theta,\tau)\, d\theta dt_0 = \int_{\mathbb{S}^2} g(\theta, t - \theta \cdot x)\, d\theta$$

The time domain Herglotz operator $\mathcal{H}g := v_g|_{\partial D \times \mathbb{R}}$.

Time-Domain Integral Operators

- The single-layer potentials causal solutions to the wave equation

$$(SL\psi)(x,t) := \int_{\partial D} \frac{\psi(x_0, t - |x - x_0|)}{4\pi |x - x_0|}\, dx_0 \qquad \psi \in C_0^\infty(\mathbb{R}; C^\infty(\partial D))$$

  We call $S\psi := SL\psi|_{\partial D, \mathbb{R}}$.

- For $h \in H_\sigma^m(\mathbb{R}_{>T}; H^{1/2}(\partial D))$ there is a unique solution

$$u = SL(S^{-1}h) \in H_\sigma^{m-3/2}(\mathbb{R}_{>T}; H^1_{loc}(\mathbb{R} \setminus \overline{D}))$$

  of the initial-boundary value problem with initial data h.

- It can be shown that the far field pattern $(SL\psi)_\infty := \mathcal{R}\psi$ where

$$\mathcal{R}\psi = \frac{1}{4\pi} \int_{\partial D} \psi(x_0, t + \xi \cdot x_0)\, dx_0, \qquad \xi \in \mathbb{S}^2,\ t \in \mathbb{R}$$

Factorization of Far Field Operator

$$F = -\mathcal{R}S^{-1}\mathcal{H}$$

We must work with Gelfand triples

$$H_\sigma^m(\mathbb{R}_{>T}; X)) \subset L_\sigma^2((\mathbb{R}_{>T}; H) \subset \tilde{H}_\sigma^{-m}(\mathbb{R}_{>T}; X^*),$$

where $X \subset H \subset X^*$ a Gelfand triple with respect to duality

$$\langle f, g\rangle = \int_{-\infty+i\sigma}^{\infty+i\sigma} \langle \mathcal{L}[g](s), \mathcal{L}[f](s)\rangle_{X^*,X}\, ds = \int_{-\infty}^{\infty} e^{-2\sigma t} \langle g(t), f(t)\rangle_{X^*,X}\, dt$$

$F : H_\sigma^{5/2}(\mathbb{R}_{>T}; L^2(\mathbb{S}^2)) \to \tilde{H}_\sigma^{-5/2}(\mathbb{R}_{>T}; L^2(\mathbb{S}^2))$

mapping $g \mapsto Fg|_{t>T}$ is bounded

Coercivity of the Middle Operator

(A Coercivity Property - due Bamberger and Ha Duong (1986))

$$-\int_{\mathbb{R}} e^{-2\sigma t} \int_{\partial D} S^{-1}(\psi)\,\partial_t\psi\, dxdt \geq C(\sigma)\|\psi\|^2_{L^2_\sigma(\mathbb{R};H^{1/2}(\partial D))}$$

for some $C(\sigma) > 0$ such that $C(\sigma) \to 0$ as $\sigma \to 0$.

Define $\mathcal{T} := (\partial_t S^{-1} - 2\sigma S^{-1})$

$\mathcal{T} : H^{3/2}_\sigma(\mathbb{R}_{>T}; H^{1/2}(\partial D)) \to \tilde{H}^{-3/2}_\sigma(\mathbb{R}_{>T}; H^{-1/2}(\partial D))$ is coercive

$$\langle \mathcal{T}\psi, \psi\rangle_{L^2_\sigma} \geq C(\sigma)\|\psi\|^2_{L^2_\sigma(\mathbb{R}_{>T};H^{1/2}(\partial D))}$$

An Adjointness Properties

$$F = -\mathcal{R} S^{-1} \mathcal{H}$$

$4\pi\mathcal{R} = \mathcal{H}^*\psi$ the L^2- adjoint.

$$\int_{\partial D} \int_{\mathbb{R}} \mathcal{H} g \, \psi \, dt \, dx_0 = 4\pi \int_{\mathbb{S}^2} \int_{\mathbb{R}} g \, \mathcal{R}\psi \, dt \, d\theta$$

But unfortunately this does hold with respect to the L^2_σ inner product!

Recall

$$\langle f, g \rangle = \int_{-\infty+i\sigma}^{\infty+i\sigma} \langle \mathcal{L}[g](s), \mathcal{L}[f](s) \rangle_{X^*,X} \, ds = \int_{-\infty}^{\infty} e^{-2\sigma t} \langle g(t), f(t) \rangle_{X^*,X} \, dt$$

Perturbed Far Field Operator

- Take $u^i_\sigma(x,t;\theta) := \delta(t-\theta\cdot x)e^{2\sigma(\theta\cdot x)}$, $\theta\in\mathbb{S}^2$

  (causal functions but not solution to the wave equation).
- $u^\sigma_\infty(\xi, t-t_0;\theta)$ be the far field pattern of the scattered field with boundary data $h := u^i_\sigma(x,t;\theta)|_{\partial D}$.

(Perturbed Far Field Operator)

The perturbed far field operator is defined by

$$(F_\sigma g)(\xi,t) := \int_{\mathbb{R}}\int_{\mathbb{S}^2} u^\sigma_\infty(\xi, t-t_0;\theta) g(\theta, t_0)\, d\theta dt_0$$

$$4\pi F_\sigma = -\mathcal{H}^*_\sigma S^{-1}\mathcal{H}_\sigma, \qquad \text{where} \qquad \mathcal{H}_\sigma g := v^\sigma_g|_{\partial D\times\mathbb{R}}$$

$$v^\sigma_g(x,t) = \int_{\mathbb{R}}\int_{\mathbb{S}^2} \delta(t-\tau-\theta\cdot x)e^{2\sigma(\theta\cdot x)} g(\theta,\tau)\, d\theta d\tau = \int_{\mathbb{S}^2} g(\theta, t-\theta\cdot x)e^{2\sigma(\theta\cdot x)}\, d\theta$$

Perturbed Far Field Operator

$$\tilde{F}_\sigma^T := 4\pi(\partial_t F_\sigma - 2\sigma F_\sigma) = (\mathcal{H}_\sigma)^* \mathcal{T} \mathcal{H}_\sigma$$

(Symmetric Factorization)

- $\tilde{F}_\sigma + (\tilde{F}_\sigma)^* = Q_F^* Q_F$, has a positive square root Q_F
- $\tilde{F}_\sigma + (\tilde{F}_\sigma)^* = (Q_\mathcal{T} \mathcal{H}_\sigma)^* (Q_\mathcal{T} \mathcal{H}_\sigma)$,

  where $Q_\mathcal{T}$ is the positive root of $\mathcal{T} + \mathcal{T}^*$.
- the ranges of Q_F^* and $(Q_\mathcal{T} \mathcal{H}_\sigma)^*$ coincide.

The Factorization Method

Consider

$$\varphi^{\infty}_{\sigma z}(\xi, t) := \chi(t + \xi \cdot z), \quad \xi \in \mathbb{S}^2$$

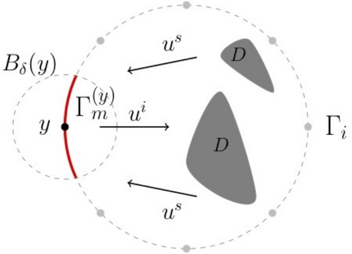
where $\chi : \mathbb{R} \to \mathbb{R}$ is smooth with compact support. Then

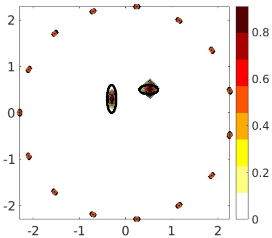
$$z \in D \Longleftrightarrow \varphi^{\infty}_{\sigma z} \in \text{Range}\left[\tilde{F}_\sigma + (\tilde{F}_\sigma)^*\right]^{1/2} = \text{Range}(\mathcal{H}_\sigma)^*$$

Although $F_\sigma \to F$ as $\sigma \to 0$ in the operator norm, to formalize a rigorous range test for the liming operator F (which is the physical measurements operator) is still an open problem.

Less Data: Quasi-backscattering, Single Wave

- Quasi-backscattering F. Cakoni, J. Rezac (2017) *JCP*





$$(\mathcal{N}_{quasi}\varphi)(t,y) = \int_{\mathbb{R}} \int_{\Gamma_m^{(y)}} u^s(x, t-\tau; y)\varphi(\tau, x) ds_x d\tau$$

- Time domain qualitative methods with one incident wave

Work in progress: F. Cakoni, G. Nakamura, J.N. Wang, M. Yamamoto