Recovery of material parameters in transversely isotropic media

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Sending waves through an object (big or small) gives a way of determining the travel times of waves between points on the boundary. Question: can we determine from this what is inside the object?

In this talk we show that in anisotropic elasticity, in the particular case of transversely isotropic media,

- qSH wave travel times $\Rightarrow$ the tilt of the axis of isotropy as well as some of the elastic material parameters, and

- the qP and qSV travel times conditionally $\Rightarrow$ a subset of the remaining parameters, in the sense if some of the remaining parameters are known, the rest are determined, or if the remaining parameters satisfy a suitable relation, they are all determined, under certain non-degeneracy conditions.

- We give a precise description of the additional issues, subject of ongoing work, that need to be resolved for a full treatment.

The problem addressed in this paper has one of its primary application in seismic tomography.

In Earth's interior, the presence of anisotropy has been widely recognized (Silver, Romanowicz and Wenk,...), including

- seismic anisotropy beneath the continents,
- anisotropy in the deep interior.

The assumption of transverse isotropy with tilted symmetry axis has played a dominant role in many studies ranging from Earth's sedimentary basins, continental dynamics and subduction, deep mantle flow and inner core.

While

- the spatially varying symmetry axis of a transversely isotropic elastic medium can be locally recovered, under certain geometric conditions,
- presently, the full recovery of elastic parameters requires some interrelationships between them.

Such relationships may be best motivated by considering models that effectively generate these parameters; these then provide possible physically, mechanically or geologically based reductions of independent parameters.

We work in an invariant setting based on Riemannian geometry
since this enables a conceptually clearer statement.

Let us start by recalling the Riemannian result, recalling that the
boundary distance function corresponds to travel times, and it
determines the lens relation (which also encodes the direction of
geodesics as the enter/exit through the boundary) on simple
manifolds, where the geodesic exponential map is a
diffeomorphism.

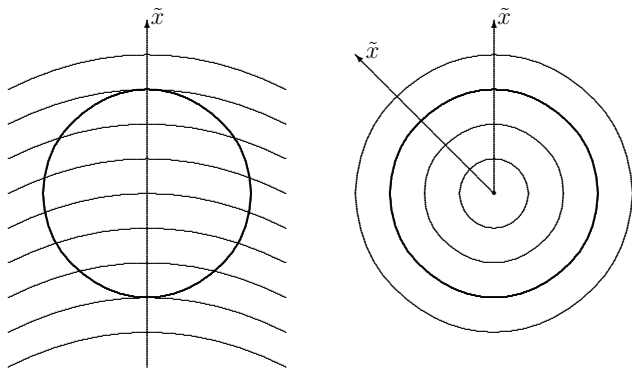### Theorem (Stefanov-Uhlmann-V., 2017)

Suppose $n \geq 3$, M compact with boundary, and let $g, \tilde{g}$ be Riemannian metrics.

Assume that ∂X is strictly convex with respect to both g and $\tilde{g}$.

Assume also that one has a function $\tilde{x}$ whose level sets are nondegenerate, strictly concave for g from the superlevel sets, and $\{\tilde{x} \geq 0\} \cap X \subset \partial X$.

If the lens relations of g and $\tilde{g}$ are the same then there exists a diffeomorphism ψ fixing ∂X such that $g = \psi^* \tilde{g}$.

Such $\tilde{x}$ exists e.g. if X is simply connected with non-positive sectional curvature, or more generally if it has no focal points.

The RHS has a degenerate level set at the origin, but is covered by a slightly modified version of the theorem... and then the assumption holds if X has non-negative curvature! (Paternain, Salo, Uhlmann, Zhou, 2016).

The result is proved by *localization*: one only needs strict convexity of ∂X and constructs $\tilde{x}$ locally.

Back to elasticity: we have a given background metric g_0, which is typically the Euclidean metric; we denote the dual metric and the dual metric function by G_0.

In general, (linearized) anisotropic elasticity is described by a system of partial differential operators, i.e. by a matrix of operators: $(a_{jk}(x, D))$.

The principal symbol, which is a tensor (matrix)-valued function on phase space (position+momentum), i.e. the cotangent bundle, is obtained by replacing all derivatives D_ℓ by the momentum variables ξ_ℓ: $(a_{jk}(x, \xi))$.

The principal symbol is non-scalar, i.e. is not a multiple of the identity map/matrix.

It can be diagonalized; the eigenvalues are the speed of the various elastic waves.

In isotropic elasticity, there are two kinds of waves, P and S waves, with:

- S waves corresponding (in spatial dimension 3) to a multiplicity 2
- and P waves a simple

eigenvalue.

In anisotropic elasticity typically the S wave eigenspace is broken up, at least in most parts of phase space.

In transversely isotropic elasticity there is a preferred axis, with respect to which the principal symbol is rotationally symmetric relative to the background metric G_0 lifted to phase space.

There are three waves then:

- the qP waves, which are the modified P-waves, as well as
- the qSV and qSH waves, with these corresponding to the 'breaking up' of the S-waves. Here V is vertical, H is horizontal, and this refers to the polarization when the anisotropy axis is vertical, in the sense that the H waves are polarized in the horizontal plane.

Of these, the qSH waves behave much like in *isotropic* elasticity in the sense that they are given by the dual metric function of a Riemannian metric, while the qP and qSV waves have a very different character.

One can parameterize transversely isotropic materials via the material constants $a_{11}, a_{13}, a_{33}, a_{55}$ and a_{66}, which are functions on the underlying position space.

In addition, there is an axis of isotropy, which can be encoded by a vector field, or better yet a one form ω.

The qSH 'energy function' (dual metric function) then depends on $a_{55}, a_{66} > 0$ and ω.

Using orthogonal coordinates relative to the metric g_0 (with G_0 the dual metric), and aligning the axis of isotropy with the third coordinate axis, possible at any given point, the wave speed of the qSH waves is given by a (squared!) Riemannian dual metric

$$G = G_{qSH} = a_{66}(x)|\xi'|^2 + a_{55}(x)\xi_3^2 = a_{66}(x)G_0 + (a_{55}(x) - a_{66}(x))\xi_3^2.$$

Isotropic case: $a_{55} = a_{66} = \mu$ (Lamé parameter).

Recall

$$G = G_{qSH} = a_{66}(x)|\xi'|^2 + a_{55}(x)\xi_3^2 = a_{66}(x)G_0 + (a_{55}(x) - a_{66}(x))\xi_3^2.$$

This, at that point, corresponds to a Riemannian metric

$$g = g_{qSH} = a_{66}(x)^{-1}|dx'|^2 + a_{55}(x)^{-1}\,dx_3^2$$
$$= a_{66}(x)^{-1}g_0 + (a_{55}(x)^{-1} - a_{66}(x)^{-1})\,dx_3^2.$$

Thus, invariantly it has the form

$$g = \alpha g_0 + (\beta - \alpha)\omega \otimes \omega,$$

i.e. the metric is a rank one perturbation of a conformal multiple of the background (say, Euclidean) metric, with $\alpha = a_{66}^{-1}, \beta = a_{55}^{-1}$ functions on the base manifold.

$\beta - \alpha$ could be incorporated into ω up to a sign, and g determines the span of ω if $\beta \neq \alpha$: the kernel of ω is well-defined (at any point in the manifold) as the 2-dimensional subspace of the tangent space restricted to which g is a constant multiple of g_0.

Under appropriate assumptions, e.g. locally near the strictly convex boundary, a Riemannian metric, g, can be recovered from its boundary distance function up to diffeomorphisms, as shown by Stefanov, Uhlmann and V. (2017).

Thus, if we know the qSH wave travel times, then in fact we know g above up to diffeomorphisms (which are the identity at the boundary).

A natural question is whether this arbitrary diffeomorphism freedom is present in our problem for the qSH wave travel times.

We show that it is *not present* under an additional condition.

An important property of a one-form, such as ω, is its integrability, or more precisely whether its kernel is an integrable hyperplane distribution, which means that $\operatorname{Ker}\omega$ is the tangent space of a smooth family of submanifolds, which are thus locally level sets of a function f, so ω is a smooth multiple of df.

In this case,

$$g = \alpha g_0 + \gamma\, df \otimes df.$$

In geological terms, this corresponds to a layered material with layers given by the level sets of f. The integrability condition is natural at least locally.

Our first theorem is:

Theorem (de Hoop-Uhlmann-V.)

Consider the class of elastic problems in which Ker ω = Ker df is an integrable hyperplane distribution on a manifold with boundary M, with ω not conormal to ∂M (so level sets of f locally intersect ∂M non-degenerately) and not orthogonal to $N^*\partial M$ relative to G_0.

Then, under the convexity conditions for Riemannian determination (up to diffeomorphisms), f, α, β are determined by the qSH travel times and the labelling of the level sets of f at the boundary.

Thus, there is *no* diffeomorphism freedom in this problem, unlike for the boundary rigidity problem in Riemannian geometry.

qSH-wave speed does not depend on $a_{11}, a_{13}, a_{33} \Rightarrow$ we need to consider qSV and qP waves.

At a point with g_0-orthogonal coordinates there and the isotropy axis aligned with the $\tilde{x}_3$ axis, the Hamiltonians for the other waves take the form ($\pm$: qP vs. qSV, and G twice the Hamiltonians)

$$G_{qP/qSV} = (a_{11} + a_{55})|\tilde{\xi}'|^2 + (a_{33} + a_{55})\tilde{\xi}_3^2$$
$$\pm \sqrt{\left((a_{11} - a_{55})|\tilde{\xi}'|^2 + (a_{33} - a_{55})\tilde{\xi}_3^2\right)^2 - 4E^2|\tilde{\xi}'|^2\tilde{\xi}_3^2},$$

where

$$E^2 = (a_{11} - a_{55})(a_{33} - a_{55}) - (a_{13} + a_{55})^2.$$

(We assume $\max\{a_{55}, a_{66}\} < \min\{a_{11}, a_{33}\}$.) In isotropic elasticity, $a_{11} = a_{33} = \lambda + 2\mu$ in terms of the Lamé parameters, while $a_{55} = \mu$, and $E = 0$.

In another coordinate systems, x_j, and corresponding dual variables ξ_j, the sound speed is given by the corresponding change of variables.

$$G_{qP/qSV} = (a_{11} + a_{55})|\tilde{\xi}'|^2 + (a_{33} + a_{55})\tilde{\xi}_3^2$$
$$\pm \sqrt{\left((a_{11} - a_{55})|\tilde{\xi}'|^2 + (a_{33} - a_{55})\tilde{\xi}_3^2\right)^2 - 4E^2|\tilde{\xi}'|^2\tilde{\xi}_3^2},$$

where

$$E^2 = (a_{11} - a_{55})(a_{33} - a_{55}) - (a_{13} + a_{55})^2.$$

Thus, these wave speeds are no longer given by a quadratic polynomial in momentum, and thus are *not* the wave speeds of a Riemannian metric *unless* $E = 0 \Rightarrow E$ measures the departure from the Riemannian, or 'elliptic', case.

The Riemannian boundary rigidity result is not applicable... but the analysis of that paper is based on the study of a class of transforms which are microlocally weighted X-ray transforms along curves, and even these general sound speeds fall in this class, with the techniques introduced by Uhlmann and V. being applicable.

Now, there are three quantities we would still like to determine: a_{11}, a_{33} and E, and we have two different wave speeds, the qSV and the qP waves that we can use. As a first step:

Theorem (de Hoop-Uhlmann-V.)

Suppose that we are given the qSH-travel time data so that ω, a_{55} and a_{66} are determined already, and assume that the hypotheses of qSH-theorem hold. Given E^2 and a_{33}, the material parameter a_{11} can be recovered from qP-travel time data.

We may instead consider a functional relationship where a_{11} determines a_{33} and E^2. This yields an alternative to the theorem above:
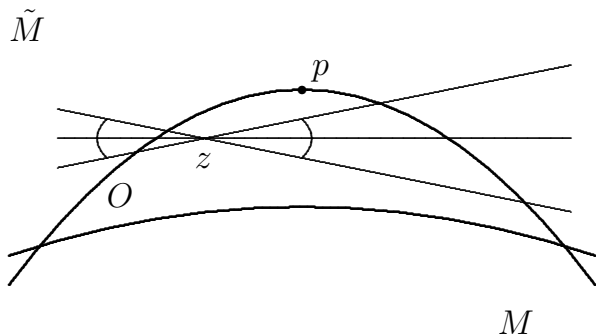

Theorem (de Hoop-Uhlmann-V.)

Suppose that we are given the qSH-travel time data so that ω, a_{55} and a_{66} are determined already, and assume that the hypotheses of Theorem 5 hold together with an additional qSV non-degeneracy condition (see later). Suppose also that we are given C^∞ functions $F : \mathbb{R} \to \mathbb{R}$ with $F' \geq 0$ and H such that $a_{33} = F(a_{11})$ and $E^2 = H(a_{11})$. Then a_{11} can be recovered from the qP- and qSV-travel time data jointly.

In order to explain the successes and limitations, let us recall the microlocal analysis background.

Following the Uhlmann-V. approach, we work with a function on M with strictly convex level sets, and localize to super-level sets of this function.

We show that the modified and localized 'normal operators' that arise from the Stefanov-Uhlmann pseudolinearization formula, which is valid for all Hamiltonian flows, are scattering pseudodifferential operators in Melrose's scattering pseudodifferential algebra, with the level set of the function at which we stop playing the role of the boundary. (Thus, this *artificial boundary* is the only one with analytic significance, while the original boundary of M simply constrains supports.)

This algebra corresponds to the Parenti-Shubin ps.d.o. algebra on
$\mathbb{R}^n$; the artificial boundary playing the role of the sphere at infinity.

Symbols: $|D_z^\alpha D_\zeta^\beta a(z, \zeta)| \leq C_{\alpha\beta} \langle z \rangle^{\ell - |\alpha|} \langle \zeta \rangle^{m - |\beta|}$.

In this algebra there are two different (and somewhat coupled) notions of ellipticity:

- that of the standard principal symbol: $|\zeta| \to \infty$ and
- that of the boundary principal symbol: $|z| \to \infty$.

The standard principal symbol corresponds to differentiable regularity, the boundary principal symbol to decay.

Ellipticity: comparable lower bounds.

Theorem (de Hoop-Uhlmann-V.)

Assume that the hypotheses of qSH Theorem hold, and that ∇f is neither parallel, nor orthogonal to the artificial boundary. (Holds near ∂M if the convex function is a perturbation of a boundary defining function for ∂M.) Assume an additional non-degeneracy condition (see later) if qSV data are used. Here convexity of the foliation always understood with respect to G_{qP}, resp. G_{qSV}, if qP, resp. qSV data are used.

Then the modified and localized 'normal operators' arising from the Stefanov-Uhlmann formula are in Melrose's scattering pseudodifferential operator algebra. Furthermore, the boundary principal symbol is elliptic at finite points (ζ bounded) for any one of E^2, a_{11}, a_{33} from the qP travel data, and for E^2 (as well as a_{11} and a_{33} if $E^2 > 0$) from the qSV travel data. Furthermore, for a_{11} from the qP-travel time data standard principal symbol ellipticity also holds.

Finally, we show the precise nature of the obstruction to full invertibility via elliptic analysis:

Theorem (de Hoop-Uhlmann-V.)

For a_{33}, E^2 from the qP or qSV travel data, as well as for E^2 and one of a_{11} and a_{33} jointly from the qP and qSV data, the standard principal symbol is not *elliptic, rather vanishes in a non-degenerate quadratic manner along the span of df at infinite points ($|\zeta| \to \infty$).*

The explanation is simple: In general, for the normal operator standard principal symbol computation at a point $\zeta \in T_x^* M$, one takes a weighted average of certain quantities evaluated at covectors for which the Hamilton vector field for the relevant sound speed is annihilated by ζ.

if $\zeta = df$ is in the axis direction, the tangent vectors involved in the integration correspond to covectors in the g_0-orthogonal plane, i.e. with vanishing $\tilde{\xi}_3$ coordinate, and there the qP and qSV sound speeds are insensitive to a_{33}, E^2 as these appear with a prefactor $\tilde{\xi}_3^2$ above.

The quadratic non-degeneracy also corresponds to this: namely the relevant coefficient is a non-degenerate multiple of $\tilde{\xi}_3^2$.

This means that the analytic framework for this inverse problem involves double characteristics, which were studied in now-classical papers of Guillemin and Uhlmann, and Mendoza and Uhlmann. However, here these need to be analyzed in the context of scattering pseudodifferential operators, and the analysis must be *global* on the manifold cut out by the artificial boundary.

Of course, we would like to determine all three of the remaining parameters ideally. One may set up a system by adding a third row and using different premultipliers, as was done in to treat boundary rigidity in the normal gauge, but one will certainly still have the double characteristic phenomena at the minimum.
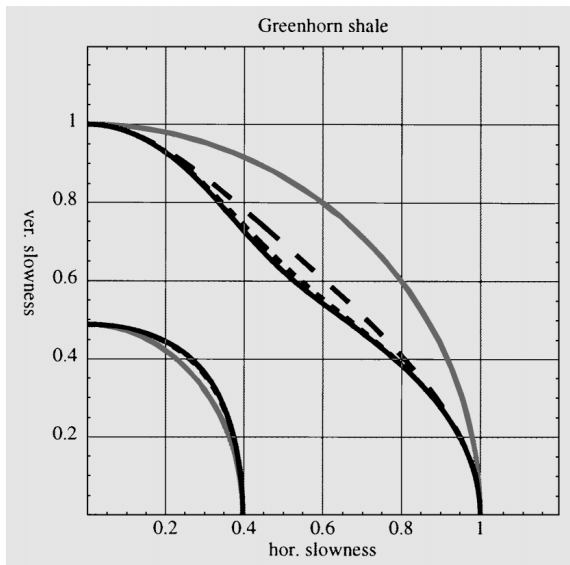
Finally: the qSV non-degeneracy condition:

Instead of using the cotangent space for parameterizing the bicharacteristics, we need to use the tangent space instead. Consider the Hamilton vector field map of the Hamiltonian function p: the tangent vector to a projected bicharacteristic $\gamma(t) = X(t)$ corresponding to the bicharacteristic $(X(t), \Xi(t))$ is $\dot{\gamma}(t) = H_{X(t)}(\Xi(t))$, where H_x is the push-forward of the Hamilton vector field to the base

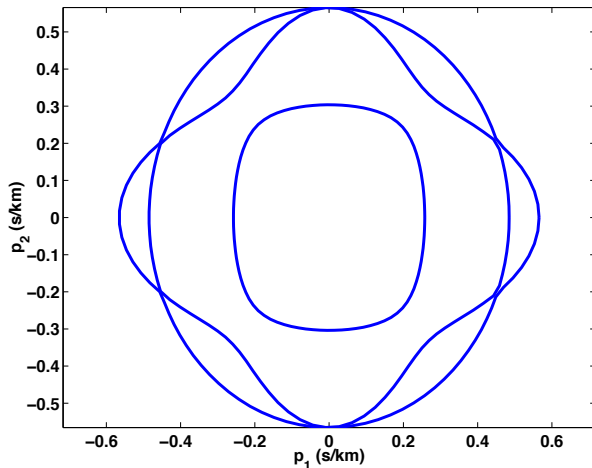$$H_x(\xi) = \sum_j \frac{\partial p}{\partial \xi_j}(x, \xi) \partial_{x_j},$$

where the notation indicates that for each base point x we consider it as a map

$$\xi \mapsto H_x(\xi).$$

This is a nonlinear map in general (though linear in Riemannian geometry.)

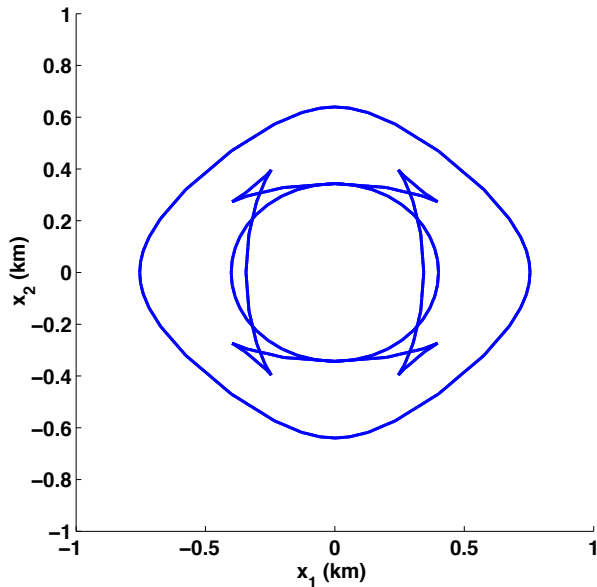The Greenhorn shale qSV wave speed level set, from Schoenberg and de Hoop.

The Greenriver shale wave speed level sets, thanks to Jianliang Qian.

In order to parameterize the bicharacteristics, $\xi \mapsto H_x(\xi)$ should be a map with a smooth inverse, at least locally along the bicharacteristics we wish to use.

This holds if DH_x is invertible. Explicitly, this differential is the Hessian matrix with ij entry $\frac{\partial^2 p}{\partial \xi_i \partial \xi_j}$. If $p(x, .)$ is a positive definite quadratic polynomial, such as in Riemannian geometry and qSH waves, then the Hessian matrix is positive definite, thus invertible. Positive definiteness of the Hessian corresponds to strict convexity of the level sets of p from the side of the sublevel sets.

In general, for interesting examples of p arising from qSV waves in transversely isotropic materials, such as for the Greenhorn shale, the strict convexity may fail.

For the qP waves it always holds.

The Greenriver shale wavefronts, thanks to Jianliang Qian.

The non-degeneracy condition relative to a convex foliation
(concave from the superlevel sets for G_{qSV}) then is:

for each point x and each vector v tangent to the convex foliation
at the point x there is a covector ξ in the cotangent space over x
such that $H_x(\xi) = v$ and the map H_x has invertible differential at
ξ, with H_x arising from G_{qSV}.

A transversely isotropic material is non-degenerate provided the
statement above holds for all v (and not just v tangent to a
particular convex foliation).

These hold in most interesting examples, and indeed always hold
for $E^2 \geq 0$ when the tangent space of the convex foliation is close
to the plane of isotropy at each point.