Seeing inside the Earth with micro Earthquakes

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Probing the Earth and the Universe with Microlocal Analysis

1. Micro Earthquakes: Four different data set
2. Finsler geometry 101
3. The inverse problem of boundary distance functions on compact Finsler manifolds
   - Tools for the proof
4. Inverse problem of boundary distance difference functions on compact Riemannian manifold
5. Reconstruction of a compact Riemannian manifold from the scattering data of internal sources

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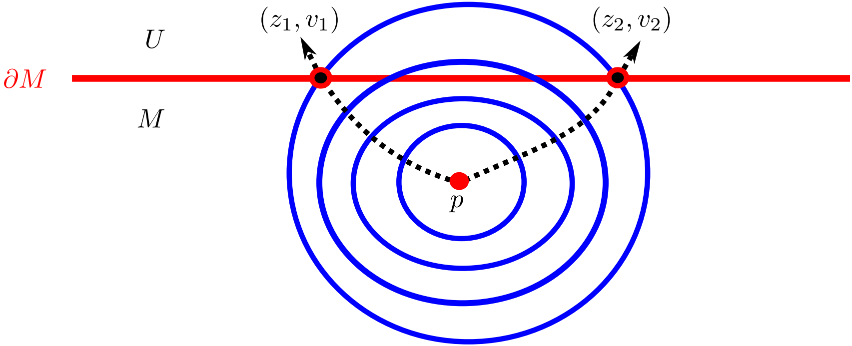
Toy model for Earthquakes

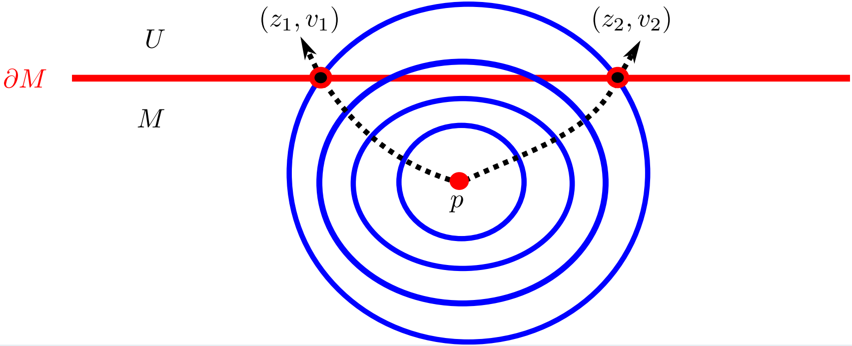
$M \subset \mathbb{R}^3$ open with smooth boundary. Denote $U := \mathbb{R}^3 \setminus M$.

Interior source acoustic wave equation

$$\begin{cases} (\partial_t^2 - c^2(x)\,\Delta_x)\,G(x,t;p,t_0) = \delta_p(x)\,\delta_{t_0}(t), & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ (p,t_0) \in M \times \mathbb{R},\ G(x,t;p,t_0) = 0, \text{ for } t < t_0,\ x \in \mathbb{R}^3. \end{cases}$$



Four different data sets related to spherical waves



Inverse problem: Recover wave speed $c(x)$ from

- travel time data (Kurylev)
- travel time difference data (Lassas-S, Ivanov)
- scattering data of internal sources $\sim$ "exit directions"
- sphere data (de Hoop-Holman-Iversen-Lassas-Ursin, de Hoop-Ilmavirta-Lassas)

It is not accurate enough to model the interior of the Earth with acoustic wave speed.

Typically one uses the elastic systems in $\mathbb{R}^3$.

Recall the talk by Joonas Ilmavirta on how the get from Elasticity to Finsler geometry.

Contents

What is a Finsler manifold?

Let M be a connected smooth manifold of dimension $n \geq 2$. We use local coordinates (x, y) for tangent bundle TM.

Let $F\colon TM \to [0, \infty)$ be a continuous function that satisfies

1. $F\colon TM \setminus \{0\} \to [0, \infty)$ is smooth.
2. for all $(x, y) \in TM$ and $a > 0$ holds $F(x, ay) = aF(x, y)$.
3. for all $(x, y) \in TM \setminus \{0\}$ the Hessian

$$\left(\frac{1}{2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}F^{2}(x, y)\right)_{i,j=1}^{n} := \Big(g_{ij}(x, y)\Big)_{i,j=1}^{n}$$

is symmetric and positive definite.

$$(2) \;\Rightarrow\; F(x, y) \neq F(x, -y)$$

$$(3) \;\Rightarrow\; F(x, y_1 + y_2) \leq F(x, y_1) + F(x, y_2)$$

and $S_xM := F^{-1}\{1\} \subset T_xM$ is convex

Pair (M, F) is called a Finsler manifold.

Riemannian and Randers metrics

Let g be a Riemannian metric and α be a 1–form then

$$F_g(x,y) := \sqrt{g_{ij}(x)\, y^i\, y^j} \quad \text{and} \quad F_\alpha(x,y) := \sqrt{g_{ij}(x)\, y^i\, y^j} + \alpha_i(x)\, y^i$$

are Finsler metrics with Hessians

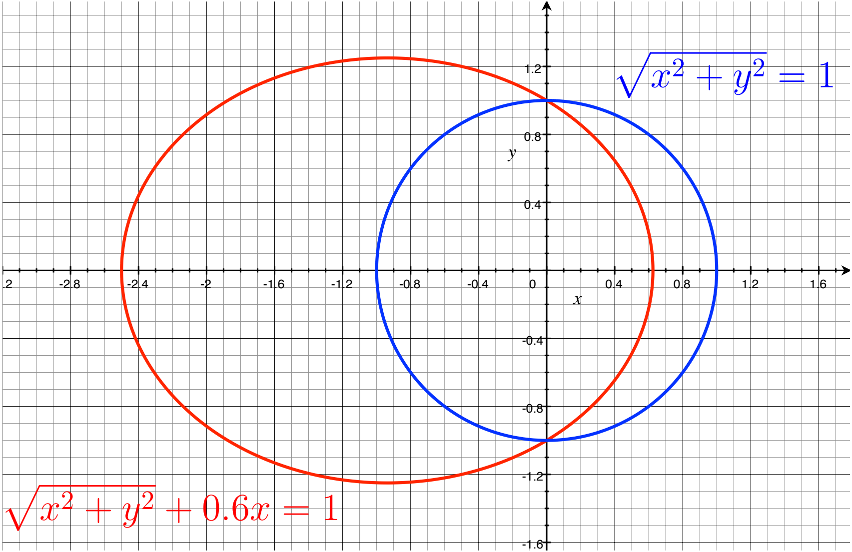
$$\frac{1}{2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}F_g^2(x,y) = g_{ij}(x)$$

$$\frac{1}{2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}F_\alpha^2(x,y) = g_{ij}(x) + \alpha_i(x)\alpha_j(x) + \frac{A_k(x)y^k}{F_g(x,y)} + \frac{B_{k\ell h}(x)y^k y^\ell y^h}{F_g^3(x,y)}$$

A Finsler function F is Riemannian if and only if

$$\frac{1}{2}\frac{\partial}{\partial y^i}\frac{\partial}{\partial y^j}F^2(x,y) = \text{ constant w.r.t. } y.$$

Euclidean and Randers unit spheres



Distance and geodesics of Finsler manifolds

Let $p, q \in M$ and let $C_{p,q}$ denote the collection of all piecewise C^1 paths from p to q.

$$d_F(p,q) := \inf \left\{ \mathcal{L}(c) := \int_0^1 F(c(t), \dot{c}(t))\, dt \,\middle|\, c \in C_{p,q} \right\}.$$

Every geodesic γ is uniquely given by the initial value $(\gamma(0), \dot{\gamma}(0)) \in TM$.

Geodesics are not preserved under change of oritentation

$$\Rightarrow \quad \gamma_{x,y}(-t) \neq \gamma_{x,-y}(t)$$

Additional differences between Riemannian and Finslerian geometries

Finsler function does not give fibervice linear duality between vectors and co-vectors. **Legendre transform**

Finsler function does not give a natural Levi-Civita connection on TM. **Chern connection**

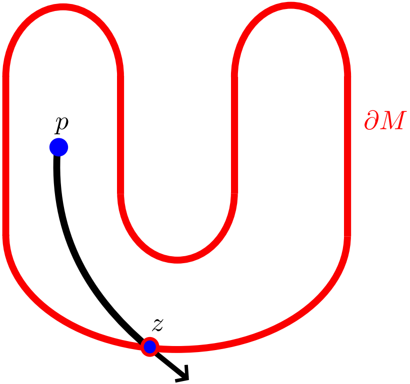
Contents

Finslerian boundary distance function

Let (M, F) be a smooth compact n-dimensional, $n \geq 2$, Finsler manifold with boundary and $p \in M^{int}$.

Boundary distance function

$$r_p \colon \partial M \to \mathbb{R}, \quad r_p(z) := d_F(p, z).$$



Direction is from p to z!

Boundary distance data

$$(\partial M, \{r_p : p \in M^{int}\})$$

Inverse problem of Finslerian boundary distance functions

Let (M_i, F_i), $i = 1, 2$ be compact smooth n-dimensional, $n \geq 2$ Finsler manifolds with boundary.

The boundary distance data of (M_1, F_1) and (M_2, F_2) agree if $\exists\, \phi\colon \partial M_1 \to \partial M_2$, diffeomorphism such that

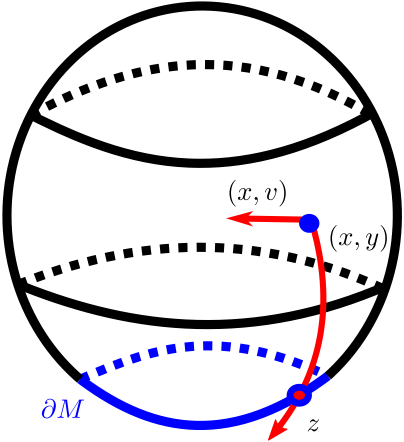
$$\left\{d_{F_1}(p, \cdot)\colon \partial M_1 \to [0, \infty) \middle| \, p \in M_1^{int}\right\} = \left\{d_{F_2}(q, \phi(\cdot))\colon \partial M_1 \to [0, \infty) \middle| \, q \in M_2^{int}\right\} \tag{1}$$

Inverse problem: Are (M_1, F_1) and (M_2, F_2) Finsler isometric if (1) holds?

Answer: Not quite!

Obstruction for the uniqueness

Define set $G(M, F)$ so that for $(x, y) \in G(M, F) \subset TM$ the geodesic $\gamma_{x,y}$ is a distance minimizer until it exits M at $z \in \partial M$.



Main theorem: Boundary distance data, optimality

Theorem (de Hoop, Ilmavirta, Lassas, S)

Let (M_i, F_i), $i = 1, 2$ *be smooth, connected, compact Finsler manifolds with smooth boundary. If the boundary distance data of* (M_1, F_1) *and* (M_2, F_2) *agree, then there is a diffeomorphism* $\Psi\colon M_1 \to M_2$ *s.t.* Ψ *on* ∂M_1 *coincides with* ϕ.

The sets $\overline{G(M_1, F_1)}$ *and* $\overline{G(M_1, \Psi^* F_2)}$ *coincide and in this set* $F_1 = \Psi^* F_2$.

For any $(x, y) \in TM_1^{int} \setminus \overline{G(M_1, F_1)}$ *there exists a smooth Finsler function* $F_3\colon TM_1 \to [0, \infty)$ *so that* $d_{F_1}(p, z) = d_{F_3}(p, z)$ *for all* $p \in M_1$ *and* $z \in \partial M_1$ *but* $F_1(x, v) \neq F_3(x, v)$.

Main theorem: Boundary distance data, analytic metric

Theorem (de Hoop, Ilmavirta, Lassas, S)

Let (M_i, F_i), $i = 1, 2$ *be smooth, connected, compact Finsler manifolds with smooth boundary. If the boundary distance data of* (M_1, F_1) *and* (M_2, F_2) *agree, and if Finsler function* F_i *is fiberwise real analytic, then there exists a Finslerian isometry* $\Psi\colon (M_1, F_1) \to (M_2, F_2)$ *such that* Ψ *on* ∂M_1 *coincides with* ϕ*.*

Strategy of the proof

The proof of optimality result consists of three steps:

1. Reconstruction of Topology
2. Reconstruction of Smooth structure
3. Reconstruction of Finsler structure

Topology: We study the map

$$\mathcal{R}_i : M_i \to (C(\partial M_i), \|\cdot\|_\infty), \quad \mathcal{R}_i(p) := d_{F_i}(p,\cdot) : \partial M_i \to [0,\infty),\ i = 1, 2.$$

For any $p \in M_i$ the minimizer z_p of $d_{F_i}(p,\cdot)$ is connected to p with a unique geodesic normal to ∂M_i. Therefore map $\mathcal{R}_i$ is 1-to-1.

Since M_i is compact, there exists $C > 0$ such that for all $p, q \in M_i$

$$\frac{1}{C} d_{F_i}(p,q) \leq d_{F_i}(q,p) \leq C d_{F_i}(p,q) \Rightarrow \|\mathcal{R}_i(p) - \mathcal{R}_i(q)\|_\infty \overset{\Delta-ie}{\leq} C d_{F_i}(p,q).$$

Smooth structure 1

By previous slide and the data, we know that a map

$$\Psi : \mathcal{R}_2^{-1} \circ \Phi \circ \mathcal{R}_1, \quad \Phi(f) = f \circ \phi^{-1} \in C(\partial M_2), \quad f \in C(\partial M_1),$$

is a homeomorphism. We have to show that Ψ is diffeomorphism.

Boundary case
We show that the boundary normal coordinates for M_1 and M_2 agree.

> One needs to use the reversed distance $d_{F_i}(z,p)$, $z \in \partial M$, $p \in M$ and
>
> inward going boundary normal geodesics

Reversed Finsler function $\widetilde{F}(x,y) := F(x,-y)$. Then

$$d_F(p,z) = d_{\widetilde{F}}(z,p), \quad p \in M, \ z \in \partial M.$$

Smooth structure 2

Interior case
We show that for every $p \in M_1^{int}$ $\exists$ open set dense set $U \subset (\partial M)^{n-1}$ s.t. for every $U \ni (z_i)_{i=1}^{n-1}$,

$$(d_{F_1}(x, z_i))_{i=1}^{n} = (d_{F_2}(\Psi(x), \phi(z_i)))_{i=1}^{n}$$

is a coordinate map w.r.t. x variable, when x is close to p. Above z_1 is any closest boundary point to p.

To show this we must prove the following

$$\underbrace{\tau(z,\nu)}_{\text{cut distance, } \nu \text{ interior unit normal}} > \underbrace{\tau_{\partial M}(z)}_{\text{boundary cut distance}}, \quad z \in \partial M.$$

The proof differs from Riemannian case due to lack of Levi-Civita connection!

Where do the Finsler functions coincide?

Since M_1 and M_2 are diffeomorphic we can assume that $M := M_1 = M_2$ and denote $F_2 = \Psi^* F_2$ on M. Thus $d_{F_1} = d_{F_2}$ on $M \times \partial M$.

Let $p \in M^{int}$,

$$S(p) = \{z \in \partial M : d_{F_i}(\cdot, z) \text{ is smooth at } p\}.$$

For $z \in S(p)$ holds

$$d(d_{F_1}(z,\cdot))\Big|_p = d(d_{F_2}(z,\cdot))\Big|_p \quad \text{and} \quad F_i^*\left(d(d_{F_i}(z,\cdot))\Big|_p\right) = 1.$$

Let

$$\Sigma_i(p) = \{y \in T_pM_i^* : y = rd(d_{F_i}(z,\cdot))\Big|_p,\ z \in S(p),\ r > 0\}.$$

These imply

$$\Sigma_1(p) = \Sigma_2(p) \quad \text{and} \quad F_1^*(p,\cdot)\Big|_{\Sigma_1(p)} = F_2^*(p,\cdot)\Big|_{\Sigma_2(p)}$$

Finsler functions agree on $\overline{G(M, F_i)}$

Recall that for $(x, y) \in G(M, F) \subset TM$ the geodesic $\gamma_{x,y}$ is a distance minimizer from $x \in M^{int}$ to the "first" boundary point.

Important technical result:

There exists a dense set $\widehat{G}(M, F_i) \subset G(M, F_i)$: For any $(x, v) \in \widehat{G}(M, F_i)$ the distance function

$$d_{F_i}(z, \cdot), \quad z := \gamma_{x,v}(\tau_{exit}(x, v)) \in \partial M,$$

is smooth at x

Therefore

The sets $\overline{G(M_1, F_1)}$ and $\overline{G(M_1, \Psi^* F_2)}$ coincide and in this set $F_1 = \Psi^* F_2$.

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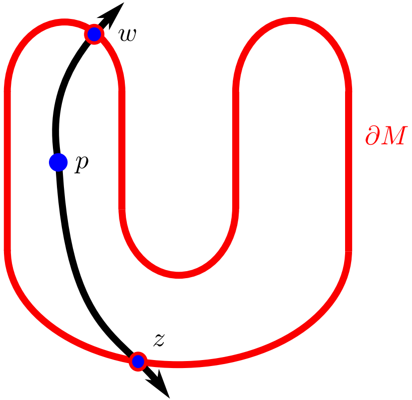
It is difficult to measure boundary distance functions



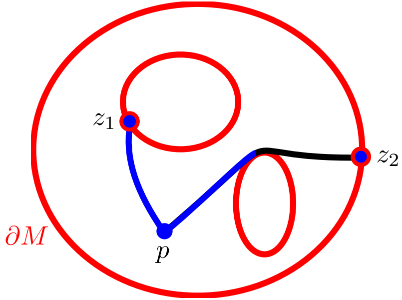
$$d(p, z) = \text{arrival time } (p \to z) - \text{origin time}$$

$$d(p, z) - d(p, w) = \text{arrival time } (p \to z) - \text{arrival time } (p \to w)$$

Boundary distance difference functions on compact Riemannian manifold

Let $n \geq 2$ and (M, g) be a n-dimensional smooth Riemannian manifold with smooth boundary ∂M. For $p \in M^{int}$ the **boundary distance difference function**, is

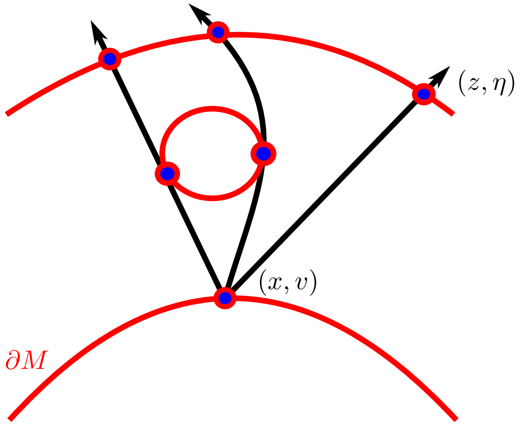
$$D_p \colon \partial M \times \partial M \to \mathbb{R}, \quad D_p(z_1, z_2) := d_g(p, z_1) - d_g(p, z_2).$$



Boundary distance difference data

$$(\partial M, \{D_p \colon \partial M \times \partial M \to \mathbb{R} \mid p \in M^{int}\}).$$

Visibility condition: Boundaries we know how to handle



- Before considered by Stefanov & Uhlmann.

- If at every point in ∂M there is a convex direction then visibility condition holds

- If $M \subset S^2$ is larger than hemisphere then ∂M does not satisfy the visibility condition

Main theorem: Boundary distance difference data

Theorem (de Hoop-S)

Let $n \geq 2$ and (M, g), be a compact, connected n–dimensional Riemannian manifold with smooth boundary ∂M which satisfies the visibility condition.

Then the boundary distance difference data determine (M, g) up to Riemannian isometry.

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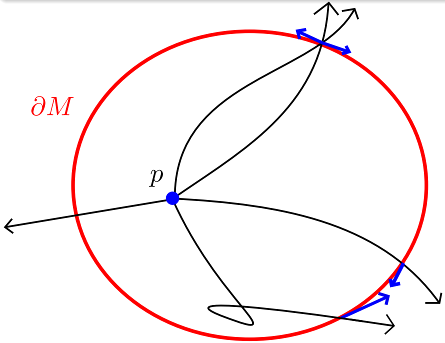
Scattering set of the interior point source

Let (M, g) be a smooth compact non-trapping Riemannian manifold with a smooth strictly convex boundary ∂M.

The **scattering set of point source** $p \in M^{int}$ is

$$R_{\partial M}(p) := \{(\gamma_{p,\xi}(\tau_{exit}(p,\xi)), (\dot{\gamma}_{p,\xi}(\tau_{exit}(p,\xi)))^T) \in T\partial M : \xi \in S_p M\}.$$



Scattering data of point sources

$$(\partial M, \{R_{\partial M}(p) : p \in M^{int}\})$$

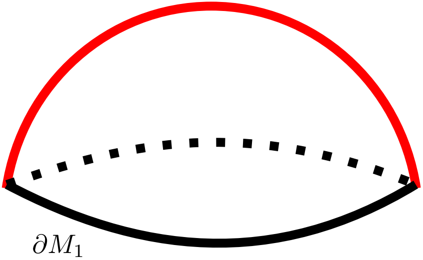
A good and a bad manifold

We denote for $x, y \in M$ and $\ell \in (0, \infty)$,

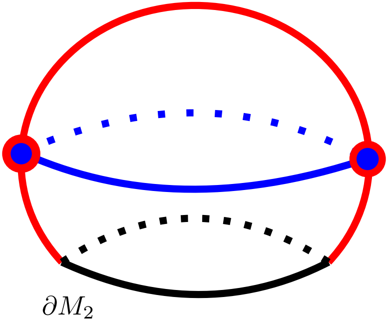
$$I(g, x, y, \ell) := \text{amount of } g\text{-geodesics of length } \ell \text{ from } x \text{ to } y.$$

Examples

M_i = polar cap in $\mathbb{R}^3$, M_1 smaller than hemisphere, M_2 larger than hemisphere.

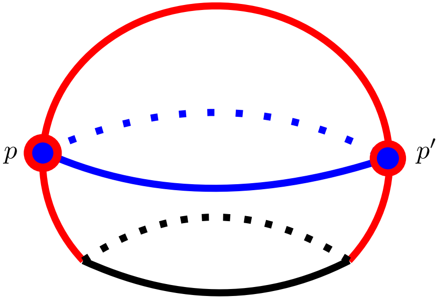


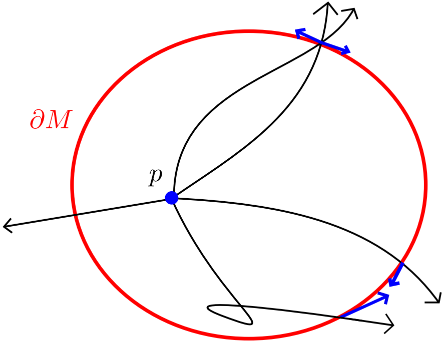
good manifold



bad manifold

Main theorem: Scattering data of internal sources





Theorem (Lassas-S-Zhou)

Let (M, g) be a smooth compact Riemannian manifold with a smooth boundary ∂M. Suppose that ∂M is strictly convex, M is non-trapping and $\sup_{x,y,\ell} I(g, x, y, \ell) < \infty$, then $\{\partial M, \{R_{\partial M}(p) : p \in M\}\}$ determine (M, g) up to isometry.

Talk was based on the following manuscripts:

I **Inverse problem for compact Finsler manifolds with the boundary distance map**, with Maarten de Hoop, Joonas Ilmavirta and Matti Lassas, preprint arXiv:1901.03902

II **Inverse problem of travel time difference functions on a compact Riemannian manifold with boundary**, with Maarten de Hoop, Journal of geometric analysis, (2018)

III **Reconstruction of a compact Riemannian manifold from the scattering data of internal sources**, with Matti Lassas and Hanming Zhou, Inverse problems and Imaging, (2018)

Thank you for your attention!

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