Determinantal Hypersurfaces, Joint Spectra, and Representations of Coxeter Groups, Part II

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Based on joint works with Z. Cuckovic, T. Peebles, M. Stessin, J. Weyman, and R. Schiffler

Let $A_0, \ldots, A_n$ be bounded linear operators on a Hilbert space V. Their **joint spectrum** is the closed set in projective space

$$\sigma(A_0, \ldots, A_n) = \\ \left\{ [x_0 : \cdots : x_n] \in \mathbb{CP}^n : \ x_0 A_0 + \cdots + x_n A_n \ \text{ not invertible} \right\}.$$

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When V is finite dimensional, the joint spectrum is given by the vanishing of the determinant

$$\mathcal{D}(x_0, \ldots, x_n) = \det \left[x_0 A_0 + \cdots + x_n A_n \right]$$

thus it has the additional structure of an algebraic subscheme of $\mathbb{CP}^n$.

More generally, given linear operators $A_0, \dots, A_n$ on a finite-dimensional vector space V over any field $\mathbb{F}$, the determinant

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is a homogeneous polynomial in $x_0, \ldots, x_n$ of degree $\dim V$.

The ideal generated by this polynomial defines an algebraic closed subscheme of projective space $\mathbb{FP}^n$ called a **determinantal hypersurface**, and we also denote it by

$$\sigma(A_0, \ldots, A_n)$$

.

A **Coxeter group** is a finitely generated group G on generators $g_1, \ldots, g_n$ defined by the following relations:

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The set of generators $\{g_1, \ldots, g_n\}$ is called a **Coxeter set of generators**, and the m_{ij}s are called the **Coxeter exponents.**

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The finite connected Coxeter groups consist of the one-parameter families A_n, B_n, D_n, and $I(n)$, and the six exceptional groups E_6, E_7, E_8, F_4, H_3, and H_4.

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Review of Part I
Coxeter groups

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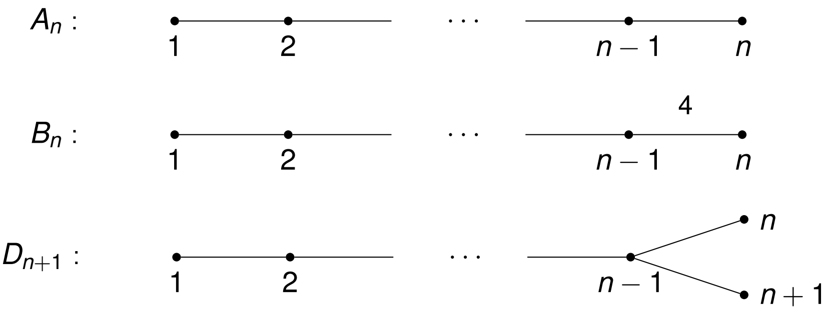

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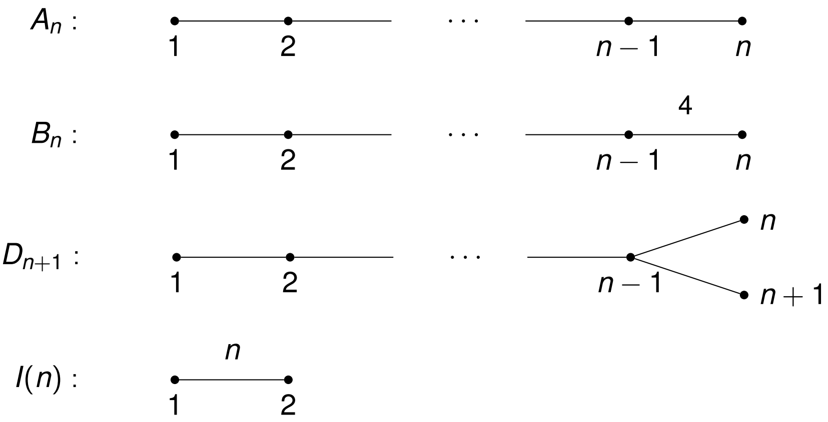

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When V is a Hilbert space we require that $GL(V)$ be the group of bounded invertible linear operators. In that case we call ρ **unitary** provided that its image consists of unitary operators.

When V is a finite-dimensional Hilbert space, the **character** of ρ is the function

$$\chi_\rho : G \longrightarrow \mathbb{C}$$

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Let $T = \{g_1, \ldots, g_n\}$ be a generating set for G. We set

$$D(T, \rho) = \sigma\big(I, \rho(g_1), \ldots, \rho(g_n)\big)$$

and refer to this set as the **joint spectrum of T on ρ**. When V is finite dimensional this is a determinantal hypersurface, and we call it the **determinant of T on ρ**

Let $\mathbb{F}$ be a field. The group G acts on the group ring $\mathbb{F}[G]$ by multiplication on the left. The resulting homomorphism

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Over $\mathbb{C}$ we have more structure. The group ring $\mathbb{C}[G]$ has inner product

$$\langle \sum_{g \in G} a_g g, \sum_{g \in G} b_g g \rangle = \sum_{g \in G} a_g \bar{b}_g$$

and corresponding induced norm

$$\left\| \sum a_g g \right\| = \sum |a_g|^2.$$

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We write $\mathbb{C}[G]^\vee$ for the Hilbert space obtained by completing with respect to this norm. (If G is finite then $\mathbb{C}[G]^\vee = \mathbb{C}[G]$.)

Left multiplication by $g \in G$ on $\mathbb{C}[G]$ induces a bounded invertible unitary linear operator $\rho(g)$ on $\mathbb{C}[G]^\vee$, and the resulting map

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If T is a generating set for G and ρ is the left regular representation of G we write just $D(T)$ instead of $D(T, \rho)$. We call $D(T)$ the **determinant of** T **on** G.

The main results

“The determinant determines the group”

Theorem (Cuckovic, Stessin, T.)

Let G be a Coxeter group with Coxeter generating set $T = \{g_1, \ldots, g_n\}$. Let G' be a group, and let $T' = \{g'_1, \ldots, g'_n\}$ be a generating set for G'.

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1. *If $D(T) \supseteq D(T')$ as subsets of $\mathbb{CP}^n$, then there is an epimorphism of groups $f : G \longrightarrow G'$ such that $f(g_i) = g'_i$ for each $1 \leq i \leq n$.*

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2. *If G is finite and $D(T) = D(T')$ as subschemes of $\mathbb{CP}^n$, then the homomorphism f from part (1) is an isomorphism.*

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“The determinant determines the representation”

Theorem (Cuckovic, Stessin, T.)

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If for two finite dimensional complex linear representations ρ_1 and ρ_2 of G we have

$$D(T, \rho_1) = D(T, \rho_2)$$

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as subschemes of $\mathbb{CP}^n$, then the representations ρ_1 and ρ_2 are equivalent.

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For $j = 2, \dots, n$ set

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It is not hard to check that $N = \langle r_1, \dots, r_n \rangle$ is an abelian normal subgroup of G and $G = B_n \ltimes N$.

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Theorem (Peebles, Stessin, T., Weyman)

With G and the elements g_i, t_i, and r_i as in the previous slide, let

$$T = \{g_2, \ldots, g_{n+1}, t_2, \ldots, t_n, r_1, \ldots, r_n, r_1^{-1}, \ldots, r_n^{-1}\}.$$

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If for two finite dimensional complex linear unitary representations ρ_1 and ρ_2 of G we have

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Let ρ be the reflection representation of the symmetric group S_n. Let $T = \{(1,2), (2,3), \ldots, (n-1, n)\}$ be the usual set of Coxeter generators for S_n.

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Then the determinant $D(T, \rho)$ is a reduced irreducible closed subscheme of $\mathbb{CP}^{n-1}$.

Similar previous results in the literature

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Dedekind

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Similar previuos results in the literature

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Almost hundred years later, it was shown by Formanek and Sibley in 1991 that the group determinant determines the isomorphism class of the group as well.

The road ahead

A guiding principle

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3. *If ρ_1 is irreducible, then $D(T, \rho_1)$ is reduced and irreducible.*

The road ahead

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6. Examples for small irreducible representations (hooks) of S_n show that in some cases one can realize the determinant of a representation as a specialization of a cluster variable. It would be very interesting to uncover the mechanism behind this phenomenon.